International Journal of
INTELLIGENT SYSTEMS AND APPLICATIONS IN ENGINEERING
ISSN:2147-6799 www.ijisae.org Original Research Paper

Bayes Estimation of Parameters of the Kibble-Bivariate Gamma Distribution Under a Precautionary Loss Function for Fuzzy Data Using Simulation

Bent AL-Huda Sahib Ghetran[1], Enas Abdul Hafedh Mohammed[2]

Submitted: 26/10/2022 **Revised:** 27/12/2022 **Accepted**: 25/01/2023

Abstract: In many real life applications, more than one variable needs to be studied. This means the need to model multivariate distributions to clarify the behavior of these variables combined and that there may be dependence among these variables. The parameters estimated according to the Bayesian method under the precautionary loss function are as close as possible to the real (hypothetical) parameters. The Bayes method under the squared loss function recorded a superiority over the Bayes method under the precautionary loss function at the cut-off coefficient (Alfa-cut = 0.3) in some simulation experiments. The greater the cutoff in the fuzzy group, the less elements that have less or equal cutoffs, and thus increase the accuracy of the estimation method.

Keywords*: Bayes estimation, Kibble-Bivariate, Fuzzy, Crisp set, Bayesian Methods*

1. Introduction

In many real life applications, more than one variable needs to be studied. This means the need to model multivariate distributions to clarify the behavior of these variables combined and that there may be dependence among these variables. The distributions are bivariate (Bivariate distributions) One of the distributions that studies the behavior of two variables that may be independent or interdependent, which is of great importance in knowing the behavior of some important phenomena. Since the estimation process depends on observations, which in many cases cannot be recorded accurately due to the errors of the experiment, personal judgment, or some unexpected situations. Then randomness and fuzziness are a mixture in it and it is expressed in fuzzy numbers, so it is necessary to generalize the traditional statistical estimation methods for real numbers into fuzzy numbers for the purpose of reaching more accurate estimates than those produced by traditional estimates.The **Gamma** distribution is considered one of the important distributions in the analysis of reliability, survival theory, and the study of various phenomena, which attracted the interest of many researchers in this field, who developed the Kama distribution with one variable, (Univariate Gamma distribution) To backward forms of binary gamma distributions to include two interrelated variables, including the binary gamma distribution for Kipple, and dual gamma distribution(Double Gamma) and the binary gamma distribution of Cheriyan and the binary gamma distribution of Gunst & Webster and the binary gamma distribution of Loaiciga & Leipnik, We'll look at one of the binary gamma distributions of Kippple, Because it provides important explanations for the variables that are interrelated with each other, as it takes into consideration the correlation between the variables in the form of a parameter that is included within its probability distribution function, which is useful in reliability analysis and survival analysis. The researcher prepares (Kipple) The first to derive the binary gamma distribution in (1941) and named after him by using two interconnected random variables, each with a Gaussian distribution, by expressing them as a binary series in a hierarchical polynomial form(Hermit polynomials) that were submitted before (Mehler ,1866),(Kipple,1941) And those are many studies and researches that dealt with the binary gamma distribution of Kipple Using the Bayesian method under a precautionary loss function assuming the fuzziness of random variables.

[1] *Kerbala University's , Faculty of Administration and Economical , Department of Statistics-Kerbala-Iraq*
[2] *Kerbala University's , Faculty of Administration and Economical , Department of Statistics-Kerbala-Iraq*
hly279392@gmail.com[1] enas.albasri@uokerbala.edu.iq[2]

Crisp set and Fuzzy set

The normal concept of the group, as it is known as (the traditional group), is that the group in which the elements either belong or do not belong to it, with absolute distinction between belonging and not belonging, with very clear and precise boundaries for each element to which it belongs, so the element is not allowed to be in the group or not. therein at the same time

(Pak & et al., 2013, 341-342)

Let it Ω be a comprehensive set, A even if it is a subset of it, each element x in A it can belong or not belong to the set A.

Let $\mu_A(x)$ it be a distinctive function for the group A that gives each element in the group Ω a degree of belonging to the group A, and this function is two-valued $\{0,1\}$, as:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

So if it was $\mu_A(x) = 1$ The element x He has complete affiliation with the set A

If it is $\mu_A(x) = 0$, then the element does not belong to the set at all A

(H. Garg et al, 2013, 397) (A. Ibrahim, A. Mohammed, 2017, 143)

Figure (1) shows the traditional group, as we note that belonging to the elements x_r and x_{r+1} equals zero for the elements و x_0 and x_1 x_2 equals one And the elements in it either belong to the group or do not belong to it.

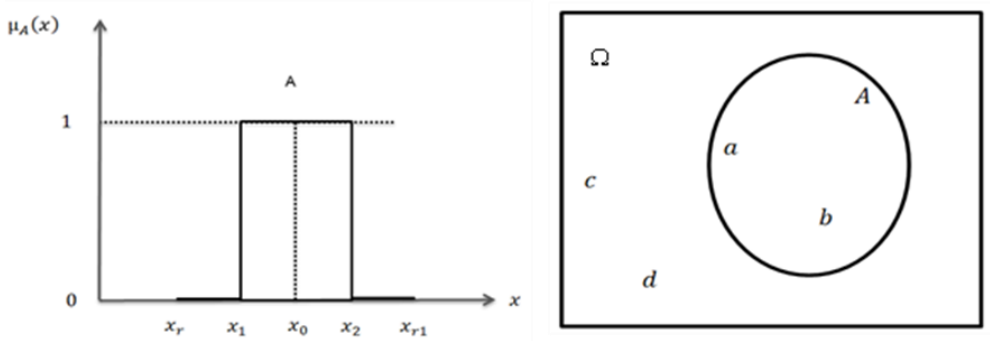

Fig 1 Graphical representation of the conventional Crisp set

Assume that $\mu_{\tilde{A}}(x_0) = 1$ then x_0 belongs completely to $\tilde{A}$ and if it is $\mu_{\tilde{A}}(x_1) = 0$, then x_1 does not completely belong to the group $\tilde{A}$ and if it is $\mu_A(x_1) = 0.8$, then x_1 belongs with a degree of 0.8 to $\tilde{A}$. If it is $\mu_{\tilde{A}}(x)$ equal to one or zero, we will get a non-fuzzy subset of the sample space (Ω).

(Danyaro & et al., 2010, 240)

Figure (2) shows the fuzzy group, as we notice that the affiliation of the elements a, c can fall between zero and one, and the element b has a degree of affiliation equal to one, and that the elements can belong to the group A with different degrees of affiliation.

It is a group whose boundaries are inaccurate, in which each element has a certain degree of belonging, through a membership function that allocates each element in the group a degree of belonging in the period [0, 1]. In which the element or object is allowed to belong to (Partial Membership)

(Pak, 2017, 504)

Let Ω be a comprehensive group, the partial fuzzy set $\tilde{\mathbf{A}}$ of Ω that is distinguished by the affiliation function $\mu_{\tilde{A}}(\mathbf{x})$, which produces between values for [0,1] each value x in the fuzzy sample space. The fuzzy group is:

$$\tilde{A} = \{(x_i, \mu_{\tilde{A}}(x_i)), x \in \Omega, i = 1,2,3, \ldots \ldots n, 0 < \mu_{\tilde{A}}(x) < 1\} \ldots .. (1)$$

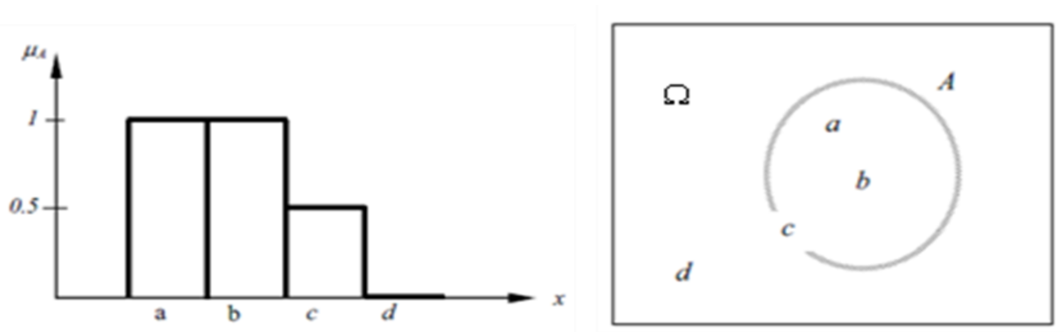

Fig. 2 Graphical representation of the ((Fuzzy set)

(Bashar, 2018, 19)

1. Membership function

It is one of the basic and important functions in fuzzy group theory, which is used to generate the belonging of elements within the fuzzy group, as it generates values within the period [0, 1] to represent the degree of belonging of each element in the traditional universal group within the (fuzzy set)

*(***Abboudi & et al, 2020, 614***)*

In other words, it is the function that draws the degree of importance of the element (the degree of affiliation) in the inclusive group to the fuzzy set, and it is a function with a positive value.

.**(Rutkowski, 2004,7-8)**

The affiliation functions are represented by a diagram whose (y-axis) represents the degree of belonging to the group, and the (x-axis) represents the normal values of the fuzzy variable. It belongs to the group and the degree of belonging to zero means that the value does not belong to the group, and the value between the two values (1,0) defines the variable degrees of belonging to the group. The affiliation function in the fuzzy group is a generalization of the characteristic function of the traditional group. Depending on the type of affiliation function, different types of fuzzy groups are obtained.

(H. Garg et al, 2013, 398).

2. α-cut

The cutoff principle in the fuzzy set was first introduced by the researcher (Zadeha,1971)

and is defined α as the lowest degree of belonging to any element in the fuzzyset $\tilde{A}$and its value falls within the

period [0 1] (H. Garg et al, 2013, 398), which represents the degree of belonging to the important elements because the important affiliation is confined between two values($a_{1,}a_m$ (on the pivot line of the fuzzy set ($Support\ \tilde{A}$) and except for those values it is Of little importance and out of scope($c$ut − out) (Auji, 2015).

3. Kibble's bivariate gamma distribution

If we have two random variables X, Y, then they are said to follow Kibble's binary gamma distribution if their joint probability density function is as follows:

$$\boldsymbol{f(x,y,v,\lambda,\beta,\rho)} = \frac{(\lambda\beta)^v}{(1-\rho)\Gamma v}\left(\frac{xy}{\rho\lambda\beta}\right)^{\frac{v-1}{2}} \mathbf{exp}\left(\frac{-\lambda x-\beta y}{1-\rho}\right) \mathbf{I}_{v-1}\left(\frac{2\sqrt{\rho\lambda\beta xy}}{1-\rho}\right) \quad \ldots \textbf{(2)}$$

$; \ \boldsymbol{x,y,v,\lambda,\beta > 0}$, $\qquad \boldsymbol{0 < \rho < 1}$

If so $\boldsymbol{\lambda = \beta = 1}$, then equation (1) becomes:

$$\boldsymbol{f(x,y,v,\rho)} = \frac{1}{\Gamma v(1-\rho)}\left(\frac{xy}{\rho}\right)^{\frac{v-1}{2}} e^{\frac{-x-y}{1-\rho}} I_{v-1}\left(\frac{2\sqrt{\rho xy}}{1-\rho}\right) \quad \ldots \textit{\textbf{(3)}}$$

$; \ \boldsymbol{x,y,v > 0}$, $\qquad \boldsymbol{0 < \rho < 1}$

It is the standard form of Kibble's binary gamma distribution

(Nadarajah & Gupta, 2006, 388)

From equation (3), the(marginal distribution) for each of X and Y is a gamma distribution with a shape parameter v ,

$X \sim gamma(\lambda , v)$

$y \sim gamma(\beta, v$

And $Corr(X,Y) = \rho$ to represent the correlation parameter between the variables X, Y.

X, Y are independent if and only if $\rho = 0$

The Kibble binary gamma distribution can be represented as a series Bessel function:

$$f(x,y,v,\lambda,\beta,\rho) = \sum_{k=0}^{\infty} f(k|\,\rho) f\left(x|v + k, \frac{\lambda}{1-\rho}\right) f\left(y|\, v + k, \frac{\beta}{1-\rho}\right) \quad \ldots \textbf{(4)}$$

Since:

$f(k|\rho)$It is the probability mass function of the negative binomial distribution$NB(V, 1-\rho)$ as follows:

$$f(k|\rho) = \frac{\Gamma(v+k)}{\Gamma(v)k!}\rho^k(1-\rho)^v \quad ; k \in Z_+ := \{0, 1, 2, \ldots\} \quad \ldots (5)$$

and that $f\left(x|v + k, \frac{\beta}{1-\rho}\right)$ and $f\left(y|\, v + k, \frac{\beta}{1-\rho}\right)$ are the probability density function of the gamma distribution with parameters $\left(v + k, \frac{\lambda}{1-\rho}\right)$ and $\left(v + k, \frac{\beta}{1-\rho}\right)$ respectively

(Tashkandy et al., *2018, 66*( )*LMoudden & Marchand*, 2020, 2)

$$\therefore \boldsymbol{F(x,y)} = \sum_{k=0}^{\infty} \frac{\boldsymbol{\rho^v(\lambda\beta)^{k+v}}}{\boldsymbol{\Gamma(k+v)k!\,\Gamma v(1-\rho)^{2k+v}}}\left(\frac{\boldsymbol{1-\rho}}{\boldsymbol{\lambda}}\right)^{\boldsymbol{k+v}} \Upsilon\left(\boldsymbol{k+v}, \frac{\boldsymbol{\lambda x}}{\boldsymbol{1-\rho}}\right)\left(\frac{\boldsymbol{1-\rho}}{\boldsymbol{\beta}}\right)^{\boldsymbol{k+v}} \Upsilon\left(\boldsymbol{k+v}, \frac{\boldsymbol{\beta y}}{\boldsymbol{1-\rho}}\right)$$

$$= \sum_{k=0}^{\infty} \frac{\rho^k (1-\rho)^v}{\Gamma(k+v)k!\Gamma v} \Upsilon\left(k+v, \frac{\lambda x}{1-\rho}\right) \Upsilon\left(k+v, \frac{\beta y}{1-\rho}\right) \quad \ldots (6)$$

Formula (6) represents the cumulative probability density function of the Kibble distribution

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+v)} [\sim NB(v,\rho)]\, \Upsilon\left(k+v, \frac{\lambda x}{1-\rho}\right) \Upsilon\left(k+v, \frac{\beta y}{1-\rho}\right) \ldots (7)$$

And formula (7) is the cumulative density function of the Kibble distribution in terms of the negative binomial distribution and in terms of the incomplete minimum quantum function.

Also, the cumulative density function of the Kibble distribution can be found in terms of the upper incomplete gamma function and the negative binomial distribution, as follows:

$$\boldsymbol{F(x,y) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+v)} [\sim NB(v,\rho)]\ \left(k+v-\Gamma\left(k+v, \frac{\lambda x}{1-\rho}\right)\right) \left(k+v-\Gamma\left(k+v, \frac{\beta y}{1-\rho}\right)\right) \ldots (8)}$$

4. Bayesian Methods

For the purpose of estimating the parameters of (Kibble's bivariate gamma) distribution

In equation (8), which is (λ, β, ρ) with fixing the parameter of the form (v), it is according to the following steps:

1. Finding the possibility function for the (Crisp data), so if we have measurements of a random sample from the distribution$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, let $X = (x_1, x_2, .., x_n)$ and $Y = (y_1, y_2, .., y_n)$, the non-blurry observed data vector be as follows:

$$l(x_i\, y_i | v, \lambda, \beta, \rho) = \prod_{i=1}^{n} f(x_i\, y_i, v, \lambda, \beta, \rho) = \frac{2^n}{(1-\rho)^{2n}(\Gamma v)^n} \left(\frac{\lambda\beta}{\rho}\right)^{n(\frac{v+1}{2})} \prod_{i=1}^{n} (x_i\, y_i)^{\frac{v-1}{2}} \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} \frac{(x_i y_i)^{2k+v-1}}{\Gamma(k+v)k!}\right) \prod_{i=1}^{n} (x_i\, y_i)^{\frac{1}{2}}$$

...*(8)*

X Drawn from X and Y drawn from Y

Where the information about can be represented by the following probability distribution:

As it x, y is seen clearly and full information is available about it.

But if it is x and y is not seen in a clear and accurate way and partial information is available about it in the form of a partial fuzzy group, then the two fuzzy groups $\tilde{x}$ and $\tilde{y}$ we get them in two steps:

$$\boldsymbol{\tilde{L}(\tilde{x}, \tilde{y}\lambda, \beta, \rho)} = \frac{2^n}{(1-\rho)^{2n}(\Gamma v)^n} \left(\frac{1}{\rho}\right)^{n(\frac{v+1}{2})} (\lambda)^{n(\frac{v+1}{2})} (\beta)^{n(\frac{v+1}{2})} \prod_{i=1}^{n} \int_0^\infty \int_0^\infty (x_i\, y_i)^{\frac{v-1}{2}} \mu_{\tilde{x}}(x) \mu_{\tilde{y}}(y)\, dy\, dx \prod_{i=1}^{n} \int_0^\infty \int_0^\infty \left(\sum_{k=0}^{\infty} \frac{(x_i y_i)^{2k+v-1}}{\Gamma(k+v)k!}\right) \mu_{\tilde{x}}(x) \mu_{\tilde{y}}(y)\, dy\, dx \prod_{i=1}^{n} \int_0^\infty \int_0^\infty (x_i\, y_i)^{\frac{1}{2}} \mu_{\tilde{x}}(x) \mu_{\tilde{y}}(y)\, dy\, dx$$

...(18)

2. Determine the joint prior probability of the parameters to be estimated as follows:

$$\boldsymbol{\pi(\underline{\theta}) = \pi_1(\lambda)\pi_2(\beta)\pi_3(\rho)} \qquad \ldots (11)$$

The previous joint distribution is written in the following form:

$$\boldsymbol{\pi}(\lambda, \beta, \rho) = \frac{\Gamma(c_3+d_3) \left(\frac{d_1}{1-\rho}\right)^{c_1} \left(\frac{d_2}{1-\rho}\right)^{c_2}}{\Gamma(c_1)\Gamma(c_2)\Gamma(c_3)\Gamma(d_3)} \boldsymbol{\lambda}^{c1-1} e^{-\frac{\lambda d_1}{1-\rho}} \boldsymbol{\beta}^{c2-1} e^{-\frac{\beta d_2}{1-\rho}} \rho^{c3-1}(1-\rho)^{d3-1} \qquad \ldots (12)$$

3. Finding the post joint probability using the inverse Bayes formula as follows:

$$\tilde{h}(\lambda, \beta, \rho | x_i\, y_i) = \frac{\rho^{-n(\frac{v+1}{2})+c_3-1} (1-\rho)^{-c1-c2+2n+d_3-1} \lambda^{c_1+n(\frac{v+1}{2})-1} \beta^{c_2+n(\frac{v+1}{2})-1} e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)} S_2 S_3 S_4}{\int_0^\infty \int_0^\infty \int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3-1} (1-\rho)^{-c1-c2+2n+d_3-1} \lambda^{c_1+n(\frac{v+1}{2})-1} \beta^{c_2+n(\frac{v+1}{2})-1} e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)} S_2 S_3 S_4\, d\rho\, d\beta\, d\lambda} \; . (13)$$

5. Precautionary Loss Function

The use of symmetric loss functions is based on the assumption that the loss is the same in any direction, but this assumption may not be fulfilled in several cases. In some cases, the positive error is more important than the negative error, and vice versa. Therefore, the use of symmetric loss functions is not appropriate. Therefore, it is preferable to use asymmetric loss functions. Among the asymmetric loss functions are the General Entropy Loss Function, the Linex Loss Function, and the DeGroot Loss Function. and a Precautionary Loss Function). The Precautionary Loss Function, which was introduced by (Nortsom, 1996), is prepared according to the following formula:

(Norstorm, 1996, 401)

$$L(\hat{\theta} - \underline{\theta}) = \frac{(\underline{\theta} - \hat{\theta})^2}{\hat{\theta}} \quad \ldots \textbf{(14)}$$

Under the precautionary loss function, underestimation is prevented because it uses the estimate around the point of origin to give conservative estimates.

(P. Mozgunov, M. Gasparini, 2019, 2)

According to the precautionary loss function, the baiz risk is given by the following formula:

$$\text{Bayes Risk} = E\left(L(\underline{\theta} - \hat{\theta})\right)$$

$$\mathbf{E}\left(\frac{(\underline{\theta}-\hat{\theta})^2}{\hat{\theta}}\right) = \int_{\forall\underline{\theta}} \frac{(\underline{\theta}-\hat{\theta})^2}{\hat{\theta}}\,\mathbf{h}\,(\underline{\theta}|\underline{t})d\underline{\theta}$$

$$= \int_{\forall\underline{\theta}} \frac{(\underline{\theta}^2-2\,\hat{\theta}\underline{\theta}+\hat{\theta}^2)}{\hat{\theta}}\,\mathbf{h}\,(\underline{\theta}|\mathbf{t})d\underline{\theta}$$

$$= \int_{\forall\underline{\theta}} \underline{\theta}^2\hat{\theta}^{-1}\mathbf{h}\,(\underline{\theta}|\underline{t})d\underline{\theta} - \int_{\forall\underline{\theta}} 2\underline{\theta}\,\mathbf{h}\,(\underline{\theta}|\underline{t})d\underline{\theta} + \int_{\forall\underline{\theta}} \hat{\theta}\,\mathbf{h}\,(\underline{\theta}|\underline{t})d\underline{\theta}$$

$$= E(\underline{\theta}^2/t)\hat{\theta}^{-1} - 2E(\underline{\theta}/t) + \hat{\theta}$$

… (15)

By deriving both sides of equation No. (15) with respect to and equating the derivative to zero, we get:

$$\frac{\partial E(\hat{\theta}-\underline{\theta})^2)}{\partial\hat{\theta}} = 0$$

$$-\hat{\theta}^{2}E(\underline{\theta}^2/t) + 1 = 0$$

$$\hat{\theta}^2_{SBBL} = E(\underline{\theta}^2|t)$$

$$\therefore \hat{\theta}_{SBBL} = \sqrt{E(\underline{\theta}^2|t)}$$

… (16)

In general, then:

$$E(\tilde{\lambda}^2/x_i y_i) = \int_0^\infty\int_0^\infty\int_0^\infty \lambda^2\tilde{h}(\lambda,\beta,\rho|x_i y_i)\,d\rho\, d\beta\, d\lambda$$

$$= \frac{\int_0^\infty\int_0^\infty\int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3-1}(1-\rho)^{-c_1-c_2+2n+d_3-1}\lambda^{c_1+n(\frac{v+1}{2})+1}\beta^{c_2+n(\frac{v+1}{2})-1}e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)}S_2S_3S_4\,d\rho\, d\beta\, d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3-1}(1-\rho)^{-c_1-c_2+2n+d_3-1}\lambda^{c_1+n(\frac{v+1}{2})-1}\beta^{c_2+n(\frac{v+1}{2})-1}e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)}S_2S_3S_4\,d\rho\, d\beta\, d\lambda} \quad \text{… (20)}$$

The Bayesian estimator for the parameters of the Kipple distribution under a precautionary loss function is as follows:

$$\therefore \hat{\lambda}_{SBPEL} = \sqrt{E(\lambda^2|t)}$$

$$= \sqrt{\frac{\int_0^\infty\int_0^\infty\int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3-1}(1-\rho)^{-c_1-c_2+2n+d_3-1}\lambda^{c_1+n(\frac{v+1}{2})+1}\beta^{c_2+n(\frac{v+1}{2})-1}e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)}S_2S_3S_4\,d\rho\, d\beta\, d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3-1}(1-\rho)^{-c_1-c_2+2n+d_3-1}\lambda^{c_1+n(\frac{v+1}{2})-1}\beta^{c_2+n(\frac{v+1}{2})-1}e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)}S_2S_3S_4\,d\rho\, d\beta\, d\lambda}} \quad \text{… (21)}$$

$$\therefore \hat{\beta}_{SBPEL} = \sqrt{E(\beta^2|t)}$$

$$= \sqrt{\frac{\int_0^\infty\int_0^\infty\int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3-1}(1-\rho)^{-c_1-c_2+2n+d_3-1}\lambda^{c_1+n(\frac{v+1}{2})-1}\beta^{c_2+n(\frac{v+1}{2})+1}e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)}S_2S_3S_4\,d\rho\, d\beta\, d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3-1}(1-\rho)^{-c_1-c_2+2n+d_3-1}\lambda^{c_1+n(\frac{v+1}{2})-1}\beta^{c_2+n(\frac{v+1}{2})-1}e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)}S_2S_3S_4\,d\rho\, d\beta\, d\lambda}} \quad \text{… (22)}$$

$$\therefore \hat{\rho}_{SBPEL} = \sqrt{E(\rho^2|t)}$$

$$= \sqrt{\frac{\int_0^\infty\int_0^\infty\int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3+1}(1-\rho)^{-c_1-c_2+2n+d_3-1}\lambda^{c_1+n(\frac{v+1}{2})-1}\beta^{c_2+n(\frac{v+1}{2})-1}e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)}S_2S_3S_4\,d\rho\, d\beta\, d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty \rho^{-n(\frac{v+1}{2})+c_3-1}(1-\rho)^{-c_1-c_2+2n+d_3-1}\lambda^{c_1+n(\frac{v+1}{2})-1}\beta^{c_2+n(\frac{v+1}{2})-1}e^{-\frac{1}{1-\rho}(\lambda d_1+\beta d_2)}S_2S_3S_4\,d\rho\, d\beta\, d\lambda}} \quad \text{…(23)}$$

$$E(u(x)|t) = \int_{\forall\underline{\theta}} u(x)h\,(\underline{\theta}|\underline{t})d\underline{\theta}$$

… **(17)**

Therefore:

$$E\,(u(\underline{\theta})) = \frac{\int_{\forall\underline{\theta}} u(\underline{\theta})(\underline{\theta}|\underline{t})\pi(\underline{\theta})d\underline{\theta}}{\int_{\forall\underline{\theta}} L(\underline{t})(\underline{\theta}|\underline{t})\pi(\underline{\theta})d\underline{\theta}}$$

… **(18)**

$u(\underline{\theta})$ Any function with parameters $\underline{\theta}$

$L(\underline{t})(\underline{\theta}|\underline{t})$ Possibility function for the current sample data

$\pi(\underline{\theta})$ The joint prior distribution of the parameters to be estimated

So if it was $u(\underline{\theta}) = \underline{\theta}^2 then$:

$$E(\underline{\theta}^2/t) = \frac{\int_{\forall\underline{\theta}} \underline{\theta}^2(\underline{\theta}|\underline{t})\pi(\underline{\theta})d\underline{\theta}}{\int_{\forall\underline{\theta}} L(\underline{t})(\underline{\theta}|\underline{t})\pi(\underline{\theta})d\underline{\theta}}$$

… **(19)**

(F. Naji & A. Rasheed, 2019, 191)

6. Fuzzy Bayesian estimator under a precautionary loss function FBPE

After extracting the subsequent joint distribution for the parameters to be estimated, we now come to find the fuzzy Bayes estimator under the quadratic loss function, as the Bayes estimator is the expected loss function for the parameter to be estimated as follows:

And that the Bayes estimations in equations (21), (22) and (23) cannot be found by the usual analytical methods, so an iterative method or approximation must be used, and Gibbs sampling will be used.

(BRÄNNSTRÖM, 2018, 18-19)

7. The experimental side

Simulation is a process of applying the user's imagination to an experimental virtual reality for the purpose of examining a specific problem or measuring a specific performance for the purpose of studying behavior and generalizing the results to the real reality. and its development over time. There have been many simulation methods, especially after the rapid development that took place in the use of electronic calculators, and because it is an effective method that enables us to manage it in a wide applied manner in practical application.

(Silva& et al. , 2010 , 429-430).

The Monte-Carlo Simulation method was adopted for the purpose of testing the efficiency of the estimation methods used to estimate the parameters of Kibble's bivariate gamma distribution using the Bayesian estimation method at quadratic and precautionary loss functions and to compare the extracted estimators using the mean squares integral error criterion. (IMSE) The sample sizes that will be worked on in selecting fuzzy groups have been determined, as a traditional sample size n = 150 has been chosen, and then the sample sizes to which the estimation methods will be applied are determined according to the cut-off factor that will be determined. And then choosing the default values for the parameters of the binary gamma distribution of Kibble, where the default values were obtained empirically from conducting several experiments and choosing the values at which the Bayes estimates stabilized and gave the best results, as shown in Table (1).

Table (1) Default values for the parameters of the binary gamma distribution for Kibble

Model / Parameter	1	2	3	4	5	6	7	8
V	1	1	1	1	1	1	1	1
Α	2	2	1	4	4.5	8	0.5	3
β	2	3	2.5	5	2	0.5	8	3
ρ	0.8	0.1	0.5	0.2	0.9	0.7	0.7	0.3

Random samples were generated following the binary gamma distribution of Kibble using a special algorithm to generate the sample of the binary gamma distribution of Kibble (General Kibble's Bivariate Gamma Distribution). Data Fuzzification (Data Fuzzification) was done by converting the traditional sample vector $\underline{\mathbf{X}} = (\underline{\mathbf{X}}_1, \ \underline{\mathbf{X}}_2)'$ from the binary gamma distributionof Kibble to fuzzy by finding the degree of belonging corresponding to each of the observations of the traditional sample vector using the trigonometric belonging function as follows:

$$\mu_A(x) = \begin{cases} 0 & if \ \ t \ < \ a \\ \frac{x-a}{b-a} & if \ \ a \le \ x \ \le b \\ 1 & if \ \ x > b \end{cases} \quad \dots (24)$$

Several cutting coefficients were chosen$\boldsymbol{\alpha - \mathrm{cut}} = \mathbf{0.3}, \mathbf{0.5}, \mathbf{0.8}$, and the reason for choosing these cutting coefficients is obtaining different types of hazy samples at each cutting level, and thus testing the effect of hazy sample sizes on the accuracy of the estimate.

Table (2) Parameters estimation and mean squares of integral error **IMSE** at traditional and fuzzy Bayesian methods at cut-off coefficients $\boldsymbol{\alpha - cut = 0.3, 0.5, 0.8}$ for all models used in the experimental side

Model		**1**		**2**		**3**		**4**		**5**	
Alfa-Cut	**Parameter**	**Estimate**	**IMSE**	**Estimate**	**IMSE**	**Estimate**	**IMSE**	**Estimate**	**IMSE**	**Estimate**	**IMSE**
0.3	$\hat{\alpha}$	**2.861**	**0.742**	**2.564**	**0.319**	**1.661**	**0.437**	**4.562**	**0.316**	***3.332***	**0.110**
	$\hat{\beta}$	**2.894**	**0.800**	**3.456**	**0.208**	**2.894**	**0.156**	**5.556**	**0.309**	**3.343**	**0.118**
	$\hat{\rho}$	**0.887**	**0.008**	**0.466**	**0.134**	**0.777**	**0.077**	**0.363**	**0.027**	**0.332**	**0.001**
0.5	$\hat{\alpha}$	**2.563**	**0.317**	**2.454**	**0.207**	**1.233**	**0.054**	**4.542**	**0.294**	**3.212**	**0.045**
	$\hat{\beta}$	**2.492**	**0.242**	**3.543**	**0.295**	**2.492**	**0.000**	**5.553**	**0.306**	**3.222**	**0.049**
	$\hat{\rho}$	**0.833**	**0.001**	**0.379**	**0.078**	**0.533**	**0.001**	**0.243**	**0.002**	**0.321**	**0.000**
0.8	$\hat{\alpha}$	**2.343**	**0.118**	**2.444**	**0.197**	**1.113**	**0.013**	**4.113**	**0.013**	**3.112**	**0.013**
	$\hat{\beta}$	**2.292**	**0.086**	**3.435**	**0.190**	**2.522**	**0.000**	**5.221**	**0.049**	**3.211**	**0.045**
	$\hat{\rho}$	**0.827**	**0.001**	**0.211**	**0.012**	**0.527**	**0.001**	**0.211**	**0.000**	**0.321**	**0.000**

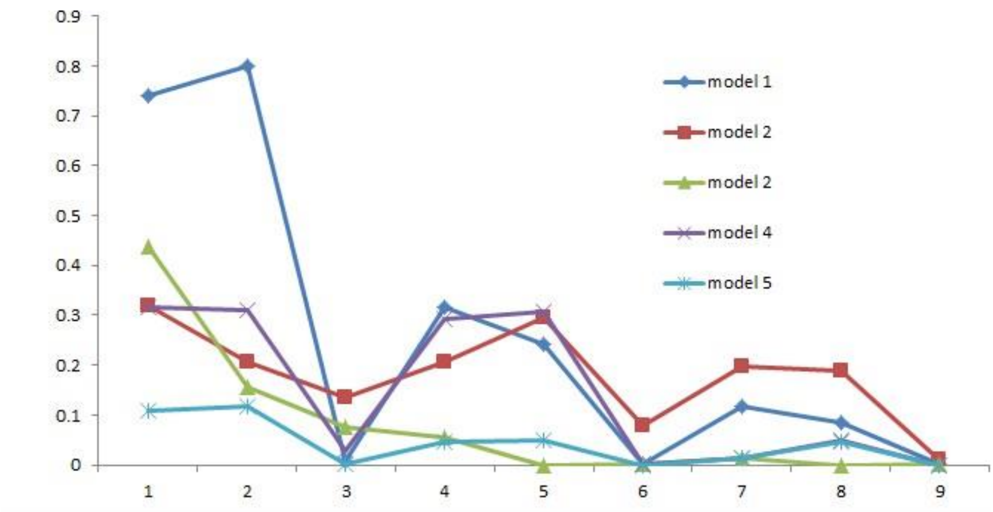

Fig 3 Mean squares integral error for the models used in the experimental side

8. Discuss the Results

1. The parameters estimated according to the Bayesian method under the precautionary loss function are as close as possible to the real (hypothetical) parameters.

2. The Bayes method under the squared loss function recorded a superiority over the Bayes method under the precautionary loss function at the cut-off coefficient (Alfa-cut = 0.3) in some simulation experiments.

3. The greater the cutoff in the fuzzy group, the less elements that have less or equal cutoffs, and thus increase the accuracy of the estimation method.

References

[1] Bashar Khaled Ali, (2018), “Selecting the best estimate of the fuzzy reliability of the Fregit distribution,” an unpublished master's thesis, University of Karbala, College of Administration and Economics

[2] Kim , Bara, Kim ,Jeongsim , (2017), " The maximum distribution of Kibble’s bivariate gamma random vector ", Operations Research Letters 45 (2017) 392–396

[3] W.F. Kibble (1941) A two variate gamma type distribution, *Sankhya:The*

[4] *Indian journal of statistics*,5(2):137-150.

[5] F. Naji , Loaiy ; A. Rasheed, Huda. (2019), "Bayesian Estimation for Two Parameters of Gamma Distribution Under Precautionary Loss Function", *Ibn Al-Haitham Jour.for Pure&Appl.Sci. IHJPAS.* , **h t t p s : / / d o i. o r g /10.30526/32.1.1914** *Vol. 32 (1) 2019*

[6] Garg , Harish , Sharma, S.P. & Rani ,Monica, (2013)," Weibull fuzzy probability distribution for analyzing the behavior of pulping unit in a paper industry" . Int. J. Industrial and Systems Engineering, Vol. 14, No. 4 , pp 395-413

[7] S. N. Sivanandam, S. Sumathi & S. N. Deepa, (2007), "Introduction to Fuzzy Logic using MATLAB", "With 304 Figures and 37 Tables", © Springer-Verlag Berlin Heidelberg.

[8] Pak ,Abbas ; (2017)," Statistical inference for the parameter of Lindley distribution based on fuzzy data" Brazilian Journal of Probability and Statistics, Vol. 31, No. 3, 502–515

[9] Howson, C. and Urbach, P. (2005). Scientific Reasoning: the Bayesian Approach .3rd edition, Open Court Publishing Company. ISBN 978-0-8126-9578-6.