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Estimation of the Fuzzy Reliability of a Mixed Distribution (Weibull-Rayleigh)

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Abstract: - The Weibull-Rayleigh distribution is a proposed mixed distribution that combines the Weibull and Rayleigh single distributions using a parameter known as the mixing ratio parameter. It is distinguished by flexibility, efficiency, and preference. This research identified a modern idea, which is the proposed new distribution (Weibull-Rayleigh), about the unusual distributions in the data visualization. In addition, failure times, which are primarily random with some fuzzy elements mixed in and expressed as fuzzy numbers, were studied. In this research, the fuzzy reliability function had been estimated for these failure times under specific ranges of affiliation to fuzzy groups, and the statistical measure of average square error was used to compare the reliability estimation techniques.

The Weibull and Rayleigh distributions, which are single distributions, are the distributions which have been testeted by using the mixing parameter. This led to the creation of the mixed distribution (Weibull -Rayleigh), from which the best estimate of fuzzy dependability was selected after using the three estimation methods, the method of greatest possibility, the method of moments, and the method of partial estimates.

Keywords: *Weibull-Rayleigh; fuzzy reliability; Estimation work*

1-1 **Introduction:**

The study of individual statistical distributions has received significant attention in statistical studies and research because of the significance that statistical distributions hold in describing the statistical behavior of observations (data), particularly when the data is homogeneous and devoid of outliers, meaning that it takes the form of a particular probability distribution but in real-world experiences (medical, engineering, scientific, etc.)

The use of mixed statistical distributions can be split into two categories: direct application and indirect application. Direct application refers to the division of the entire population into (R) of partial communities. Every sub-community has a statistical distribution for all other partial communities with different parameters, or it could be a member of a family of distributions with various characteristics. In terms of the indirect application, it aims to employ the (Mixture distribution) as a sophisticated statistical technique to achieve great flexibility in statistical analysis as well as in cluster analysis.

From a statistical perspective, reliability is the probability that the device or machine works to accomplish a specific task for a period of time until the malfunction occurs in this machine, but when this malfunction will occur and when it does, how long will it take to fix it? This is because the reliability theory (Reliability Therom) is the possibility of the device or machine being able to complete operations without failure (malfunction). Here, we observe that the presence of ambiguity results in an inaccuracy in determining the timing of the machine's breakdown. The optimal decision-making procedure to solve any problem is impacted by the ambiguity. Most of the issues that researchers face in real life may include

1-2 Fuzzyiness:

In a more specific sense, ambiguity in characterizing objects is what is meant by the term "fuzziness," which is defined as a condition of uncertainty. Moreover, it is a notion for describing ambiguity in an event and quantifying its level. When this occurs, time transitions from being a

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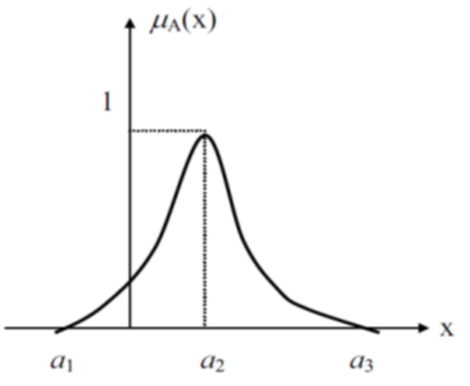
component of a group to not being a component of a group, turning all regular times into fuzzy ones.

1-3 Fuzzy Numbers :

Fuzzy numbers are always triangular or trapezoidal numbers or any other shape, and they are used to the shape(1-2) Shows the fuzzy figure
depict the state of ambiguity and certainty that goes along with particular observations. The curve of fuzzy numbers is depicted in Figure (1-2) below for various values.



1-4 Fuzzy Reliability:
The likelihood that a unit or device will continue to function after a certain amount of time (t) has passed since use is known as reliability. If T is a continuous random variable with a value greater than 0, the reliability function $R_T(t)$can be written as follows:

$$R_T(t) = P(T \geq t) = \int_t^{\infty} f_T(t) d_x \quad \dots\dots (2-5)$$

It is a monotonous function that is inversely proportional to time, which indicates that the reliability function's value decreases as the lifespan of the device increases, which means that:

$$R(t_1) > R(t_2) > R(t_3) > R(t_4) > \cdots > Rt_{\infty}$$

As a result, the reliability function has a value of (1) at the beginning of the machine or device (zero time) and then gradually decreases until the maximum duration of the machine or device's life, where its value reached zero, i, e :

$R(t = 0) = 1$

$R(t = Max(t)) = 0$

When the dependability value is equal to 1, it means that the machine or device is just starting to work and will continue to do so until time runs out. If the reliability value is equal to (zero), the machine or device does not work (t). As a result, the reliability function may be used to express the chance of failure at time interval (t_1, t_2) at time (t) as follows: time (t). Therefore, the probability of failure in the time t_1, t_2 can be expressed by the reliability function as follows:

$$R_T(t) = R_T(t_1) - R_T(t_2) \quad \dots\dots (2-6)$$

The symbol $\lambda(t)$ stands for the failure rate during the period (t_1, t_2), which can be expressed as follows:

$$\lambda(t) = \frac{R_T(t_1) - R_T(t_2)}{(t_1 - t_2)R(t)}$$

or:

$$\lambda(t) = \frac{R(t) - R(t + \Delta t)}{\Delta t R(t)} \quad \dots\dots (2-7)$$

As a result, we may state that fuzzy dependability is denoted by, which is the fuzzy reliability function, and denotes the likelihood that the machine or device will accomplish the job expected to do for a certain amount of time with varied degrees of success under typical circumstances.

So, $\mu_{\tilde{A}_i}(R)$ represents the degree of affiliation in $\tilde{A}_i$, that is:

$$\check{R}(t) = \mu_{\tilde{A}_i}(R). R(t) \quad \dots\dots (2-8)$$

Since the:

$$R(t) = \int_t^{\infty} f(t) dt$$

Therefore:

$$\check{R}(t) = \int_{t}^{\infty} f(t)dt.\mu_{\tilde{A}_i}(R) \quad \ldots\ldots(2-9)$$

: 1-5 Weibull Distribution

The density function of the Weibull distribution can be expressed as follows:

$$f(t,\lambda,\delta) = \lambda\delta t^{\delta-1}e^{-\lambda t^{\delta}} \qquad t > 0\,;\ \lambda,\delta > 0 \qquad \ldots\ldots(2-10)$$

Where as

λ: Scale Parameter

δ :Shape Parameter (That is, to determine the shape of the statistical distribution)

Where as, the aggregate function of the Weibull distribution (CDF) is:

$$F(t,\lambda,\delta) = \int_0^t \lambda\delta u^{\delta-1}e^{-\lambda u^{\delta}}$$

$$F(t,\lambda,\delta) = 1 - e^{-\lambda t^{\delta}} \qquad \ldots\ldots(2-11)$$

Thus, the reliability function of the Weibull distribution is as follows:

R(t, λ,δ)=1-F(t)

That is:

$$R(t,\lambda,\delta) = e^{-\lambda t^{\delta}} \qquad \ldots\ldots(2-12)$$

1-6 Rayleigh distributoin:

The Rayleigh distribution density function can be expressed as follows:

$$f(t,,\theta) = 2\theta t e^{-\theta t^2} \qquad t > 0\ ,\theta > 0 \qquad \ldots\ldots(2-13)$$

Where:

θ :Scale Parameter

Whereas, the aggregate function of the Rayleigh distribution is as follows:

$$F(t,\theta) = \int_0^t 2\theta u\ e^{-\theta u^2}du$$

That is:

$$F(t,\theta) = 1 - e^{-\theta t^2} \qquad \ldots\ldots(2-14)$$

Thus, the reliability function for the Rayleigh distribution is as follows:

R(t, θ)=1-F(t)

$$R(t,\theta) = e^{-\theta t^2} \qquad \ldots\ldots(2-15)$$

1-7 Mixture New Distribution (Weibull – Rayleigh):

Based on equation (2-15), the suggested new distribution is a mixture of two distributions (Weibull-Rayleigh), a Weibull distribution with a measurement parameter and a shape parameter, and a Rayleigh distribution with a single parameter, the shape parameter. The mixture formula is as follows:

$$f(t) = Zf_1(xt) + (1-Z)f_2(t) \qquad \ldots\ldots(2-19)$$

where:

$$Z = \frac{\alpha}{\alpha+1}$$

The percentage contribution of each of the component distributions to the mixture distribution is thought to represent the parameter. Thus, the following is the probability density function for the mixture distribution:

$$f(\alpha,\delta,\lambda,\theta,t) = \left(\frac{\alpha}{\alpha+1}\right)\lambda\delta t^{\delta-1}e^{-\lambda t^{\delta}} + \left(\frac{1}{\alpha+1}\right)2\theta t.e^{-\theta t^2} \qquad \ldots\ldots(2-20)$$

where:

$\delta,\lambda,\theta,x > 0, \alpha > -1$

λ :Scale Parameter

δ : Shape Parameter

θ :Scale Parameter

α : Mixing proportion parameter

Where the probability density function is satisfied by the following two conditions:

$f(t) \geq 0$

$$\int_0^{\infty} f(\alpha,\delta,\lambda,\theta,t)dt = 1 \qquad \ldots\ldots(2-21)$$

Integrating equation (2-20) as:

$$f(\alpha,\delta,\lambda,\theta,t) = \int_0^{\infty} \left(\frac{\alpha}{\alpha+1}\right)\lambda\delta t^{\delta-1}e^{-\lambda t^{\delta}} + \left(\frac{1}{\alpha+1}\right)2\theta t.e^{-\theta t^2} \qquad \ldots\ldots(2-22)$$

If we take the first term and assume that m_1 is of the following form:

$$m_1 = \frac{\alpha}{\alpha+1}\int_0^{\infty} \lambda\delta t^{\delta-1}e^{-\lambda t^{\delta}}dt_1$$

Substitute:

$$U = \lambda t^{\delta},\quad t^{\delta} = \frac{u}{\lambda}\ ,\ t = (\frac{u}{\lambda})^{\frac{1}{\delta}}\ ,\ dt = \frac{1}{\lambda\delta}(\frac{u}{\lambda})^{\frac{1}{\delta}-1}du$$

Then m_1 takes the form:

$$m_1 = \frac{\alpha}{\alpha+1}\lambda\delta\int_0^{\infty}\left[(\frac{u}{\lambda})^{\frac{1}{\delta}}\right]^{\delta-1} e^{-u}\ \frac{1}{\lambda\delta}(\frac{u}{\lambda})^{\frac{1}{\delta}-1}du$$

$$m_1 = \frac{\alpha}{\alpha+1}\int_0^{\infty}(\frac{u}{\lambda})^{1-\frac{1}{\delta}+\frac{1}{\delta}-1}\ e^{-u}\ du$$

$$m_1 = \frac{\alpha}{\alpha+1}\int_0^{\infty} e^{-u}\ du$$

$$m_1 = \frac{\alpha}{\alpha+1} \quad \ldots .1$$

And if we take the second term and assume that m_2 is of the following form:

$$m_2 = \frac{1}{\alpha+1}\int_0^{\infty}\left(\frac{1}{\alpha+1}\right)2\theta t.\,e^{-\theta t^2}\quad dt$$

Substitute:

$$z = \theta t^2 \quad , t^2 = \frac{z}{\theta} \quad ,$$

$$t = (\frac{z}{\theta})^{\frac{1}{2}} \quad , \quad dt = \frac{1}{2\theta}(\frac{z}{\theta})^{\frac{-1}{2}}dz$$

Then m_2 takes the form:

$$m_2 = \frac{1}{\alpha+1}\ 2\theta\int_0^{\infty}(\frac{z}{\theta})^{\frac{1}{2}}e^{-z}\frac{1}{2\theta}(\frac{z}{\theta})^{\frac{-1}{2}}dz$$

$$m_2 = \frac{1}{\alpha+1}\int_0^{\infty} e^{-z}\ dz$$

$$m_2 = \frac{1}{(\alpha+1)} \quad \ldots .2$$

Permiting:

$$f(\alpha,\delta,\lambda,\theta,t) = \frac{\alpha}{\alpha+1} + \frac{1}{\alpha+1} = 1$$

The Figure(2-2) and Figure(2-3) display the probability density function curve at various values of the parameters of the mixed distribution, demonstrating that the requirement that the function is probabilistic and positive for all values of the random variable (t) is satisfied (Weibull-Rayleigh)

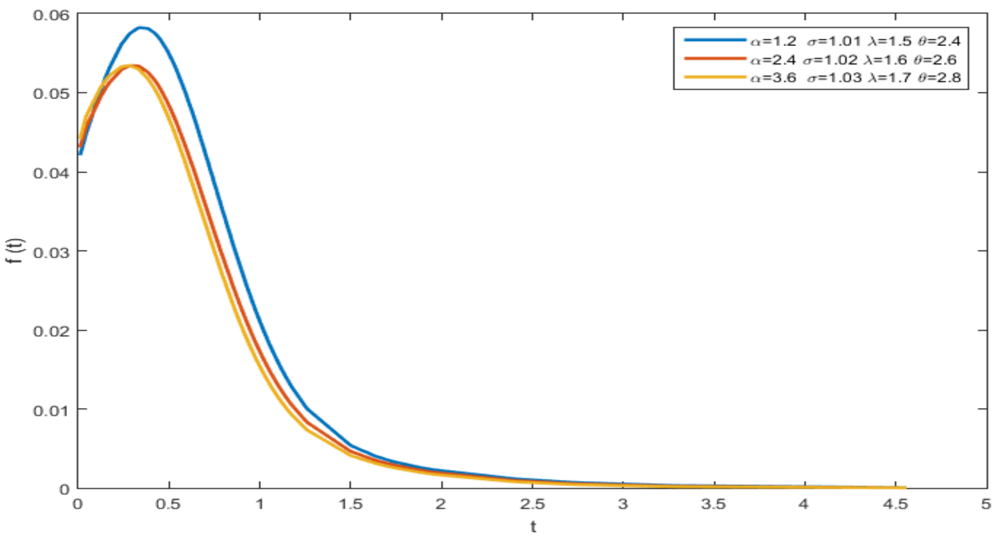


Fig (2-2): the curve of the pdf function of the mixed distribution (Weibull-Rayleigh)

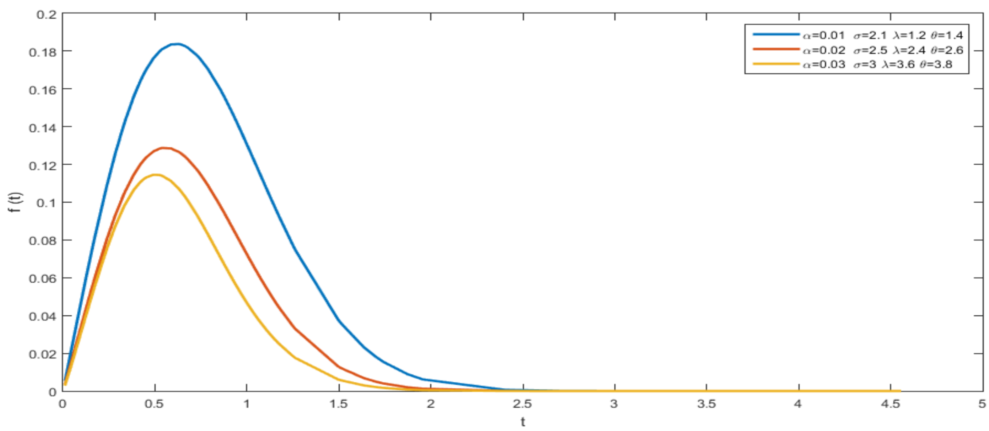


Fig (2-3): the curve of the pdf function of the mixed distribution (Weibull-Rayleigh)

Thus, the cumulative aggregate function of the new mixed distribution (Weibull-Rayleigh) is as follows:

$$F(\alpha,\delta,\lambda,\theta,t) = \left(\frac{\alpha}{\alpha+1}\right)\left(1-e^{-\lambda x^{\delta}}\right) + \left(\frac{1}{\alpha+1}\right)\left(1-e^{-\theta x^{2}}\right) \quad \dots\dots(2-23)$$

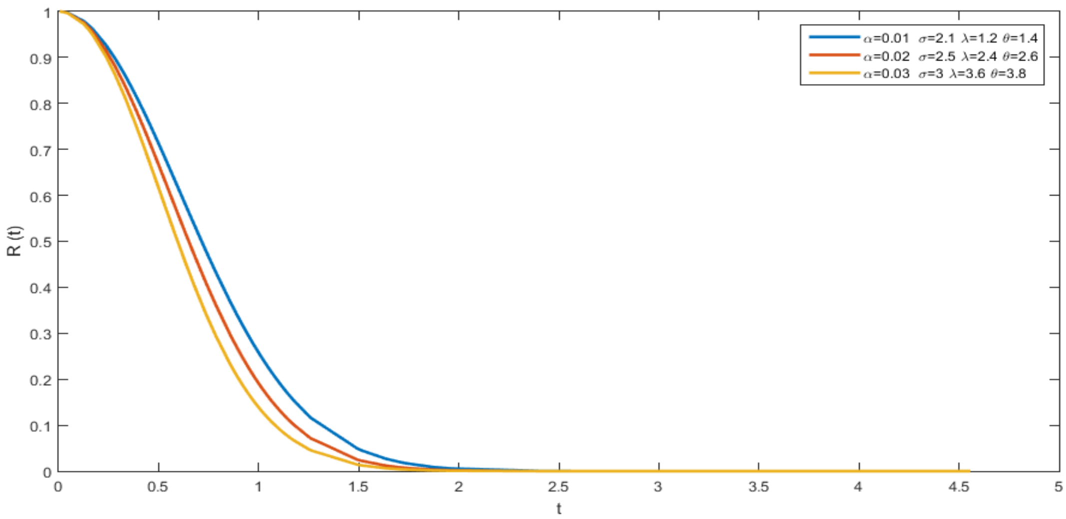
The reliability function for the new mixed distribution (Weibull-Rayleigh) is as follows:

$$R(\alpha,\delta,\lambda,\theta,t) = \left(\frac{\alpha}{\alpha+1}\right)\left(e^{-\lambda t^{\delta}}\right) + \left(\frac{1}{\alpha+1}\right)\left(e^{-\theta t^{2}}\right) \dots\dots(2-24)$$

When multiplying the random variable t by the fuzziness $\tilde{A}_i$, which represents the closed period [0,1], we get the following fuzzy reliability function:

$$\check{R}(\alpha,\delta,\lambda,\theta,t,\tilde{A}_i) = \left(\frac{\alpha}{\alpha+1}\right)\left(e^{-\lambda \tilde{A}_i t^{\delta}}\right) + \left(\frac{1}{\alpha+1}\right)\left(e^{-\theta \tilde{A}_i t^{2}}\right) \dots\dots(2-25)$$

Fig (2-4): the reliability function curve for the mixed distribution (Weibull-Rayleigh)



1-8 Estimation Methods:

Estimation methods are mathematical processes aimed at finding the estimators of the unknown population parameters according to the sample used. Here we review the following methods:

1-8-1Maximum Likelihood Method (MLE):

The maximum possibility function for the independent random variables $t_1, t_2, \dots\dots, t_n$ will be of the following form:

$$L(t_1, t_2, \dots\dots, t_n, \alpha, \delta, \lambda, \theta) = \prod_{i=1}^{n} f(t_i, \alpha, \delta, \lambda, \theta) \quad \dots\dots(2-40)$$

The formula for the mixed (Weibull-Rayleigh) distribution in (2-40) can be written as:

$$f(t;\alpha;\theta;\lambda;\delta) = \frac{e^{-t^{\delta}\lambda} t^{-1+\delta}\alpha\delta\lambda}{1+\alpha} + \frac{2\theta e^{-\theta t^2}}{(1+\alpha)}$$

So the likelihood function can be expressedas:

$$L(t_1, t_2, t_3, \dots\dots, t_n, \alpha; \theta; \lambda; \delta) = \sum\left(\frac{e^{-ti^{\delta}\lambda} ti^{-1+\delta}\alpha\delta\lambda}{1+\alpha} + \frac{2\theta e^{-\theta ti^2}}{(1+\alpha)}\right)$$

By maximizing the function and taking the natural logarithm of both sides, we get:

$$lnL(t_1, t_2, t_3, \dots\dots, t_n, \alpha; \theta; \lambda; \delta) = \sum_{i=1}^{n} ln\left(\frac{e^{-ti^{\delta}\lambda} ti^{-1+\delta}\alpha\delta\lambda}{1+\alpha} + \frac{2\theta e^{-\theta ti^2}}{(1+\alpha)}\right) \quad (2-41)$$

By partially deriving the parameters of the mixed distribution (Weibull-Rayleigh), we get the following:

First: with respect to, By talking derivative the parameter (α) using equation (2-41), we get:

$$\frac{\partial lnL}{\partial\alpha} = \sum_{i=1}^{n}\left(\frac{-\frac{2e^{-t^{2}\theta}\theta}{(1+\alpha)^{2}}-\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\alpha\delta\lambda}{(1+\alpha)^{2}}+\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\delta\lambda}{1+\alpha}}{\frac{2e^{-t^{2}\theta}\theta}{1+\alpha}+\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\alpha\delta\lambda}{1+\alpha}}\right) = 0 \quad (2-42)$$

Second: with respect to. By talking derivative the parameter (θ) using equation (2-41), we get:

$$\frac{\partial lnL}{\partial\theta} = \sum_{i=1}^{n}\left(\frac{\frac{2e^{-t^{2}\theta}}{1+\alpha}-\frac{2e^{-t^{2}\theta}t^{2}\theta}{1+\alpha}}{\frac{2e^{-t^{2}\theta}\theta}{1+\alpha}+\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\alpha\delta\lambda}{1+\alpha}}\right) = 0 \quad (2-43)$$

Third: with respect to. By talking derivative the parameter (δ) Using Equation (2-41), we get:

$$\frac{\partial lnL}{\partial\delta} = -\sum_{i=1}^{n}\left(\frac{\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\alpha\lambda}{1+\alpha}+\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\alpha\delta\lambda Ln[t]}{1+\alpha}-\frac{e^{-t^{\delta}\lambda}t^{-1+2\delta}\alpha\delta\lambda^{2}Ln[t]}{1+\alpha}}{\frac{2e^{-t^{2}\theta}\theta}{1+\alpha}+\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\alpha\delta\lambda}{1+\alpha}}\right) = 0 \quad (2-44)$$

Fourth: with respect to. By talking derivative the parameter (2-41) using the equation (λ), we get:

$$\frac{\partial lnL}{\partial\lambda} = -\sum_{i=1}^{n}\left(\frac{\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\alpha\delta}{1+\alpha}-\frac{e^{-t^{\delta}\lambda}t^{-1+2\delta}\alpha\delta\lambda}{1+\alpha}}{\frac{2e^{-t^{2}\theta}\theta}{1+\alpha}+\frac{e^{-t^{\delta}\lambda}t^{-1+\delta}\alpha\delta\lambda}{1+\alpha}}\right) = 0 \quad (2-45)$$

From equations (2-42), (2-43), (2-44), (2-45) and using one of the numerical iterative methods, we obtain the greatest possible estimators for the mixed distribution (Weibull-Rayleigh) using the (f.solve) method, so we get the estimators $(\hat{\alpha}_{MLE}, \hat{\lambda}_{MLE}, \hat{\delta}_{MLE,}, \hat{\theta}_{MLE})$. substituting the estimators into equation (2-25), we obtain the fuzzy reliability estimators for this method:

$$\hat{R}_{MLE}(t) = \left(\frac{\hat{\alpha}_{MLE}}{\hat{\alpha}_{MLE}+1}\right)\left(e^{-\hat{\lambda}_{MLE}\tilde{A}_{i}t^{\hat{\delta}_{MLE}}}\right)+\left(\frac{1}{\hat{\alpha}_{MLE}+1}\right)\left(e^{-\hat{\theta}_{MLE}\tilde{A}_{i}t^{2}}\right) \quad (2-46)$$

1-8-2 Method of Moments:

This method is based on estimating the (unknown) community moments in terms of the (known) sample moments, by equating the sample moments with the community and knowledge moments as follows:

$$m_r = \frac{\sum_{i=1}^{n} t_i^{r}}{n}$$
$$\mu_r = E(t) \quad \ldots(2)$$
$$m_r = \mu_r \quad \ldots..(3)$$

So we get four equations for the sample moments:

$$m_1 = \frac{\sum_{i=1}^{n} t_i}{n} \quad (1)$$
$$m_2 = \frac{\sum_{i=1}^{n} t_i^{2}}{n} \quad (2)$$
$$m_3 = \frac{\sum_{i=1}^{n} t_i^{3}}{n} \quad (3)$$
$$m_4 = \frac{\sum_{i=1}^{n} t_i^{4}}{n} \quad (4)$$

According to the distribution of the new (W-R) and equating the moments of the sample with the moments of the population, the equations are as follows:

$$\frac{\alpha}{(\alpha+1)\lambda^{\frac{1}{\delta}}}\Gamma\left(\frac{1}{\delta}+1\right)+\frac{1}{(\alpha+1)\theta^{\frac{1}{2}}}\frac{1}{2}\sqrt{\pi} = \frac{\sum_{i=1}^{n} t_i}{n} \quad (2-47)$$

$$\frac{\alpha}{(\alpha+1)\lambda^{\frac{2}{\delta}}}\,\Gamma\left(\frac{2}{\delta}+1\right)+\frac{1}{(\alpha+1)\theta} = \frac{\sum_{i=1}^{n} t_i^{\,2}}{n} \qquad (2-48)$$

$$\frac{\alpha}{(\alpha+1)\lambda^{\frac{3}{\delta}}}\,\Gamma\left(\frac{3}{\delta}+1\right)+\frac{1}{(\alpha+1)\theta^{\frac{3}{2}}}\,\frac{3}{4}\sqrt{\pi} = \frac{\sum_{i=1}^{n} t_i^{\,3}}{n} \quad (2-49)$$

$$\frac{\alpha}{(\alpha+1)\lambda^{\frac{4}{\delta}}}\,\Gamma\left(\frac{4}{\delta}+1\right)+\frac{1}{(\alpha+1)\theta^{2}}\,\frac{15}{8}\sqrt{\pi} = \frac{\sum_{i=1}^{n} t_i^{\,4}}{n} \quad (2-50)$$

By equating the sample moments with the community moments of equations (2-47), (2-48), (2-49), (2-50), we obtain a nonlinear system of equations, and using the (f.solve) method, we obtain the estimators of the moments $(\hat{\alpha}_{mom}, \hat{\lambda}_{mom}, \hat{\delta}_{mom,}, \hat{\theta}_{mom})$ substituting the estimators into equation (2-22), we obtain the fuzzy reliability estimators for this method:

$$\hat{R}_{mom}(t) = \left(\frac{\hat{\alpha}_{mom}}{\hat{\alpha}_{mom}+1}\right)\left(e^{-\hat{\lambda}_{mom}\tilde{A}_i t^{\hat{\delta}_{mom}}}\right) + \left(\frac{1}{\hat{\alpha}_{mom}+1}\right)\left(e^{-\hat{\theta}_{mom}\tilde{A}_i t^{2}}\right) \ldots (2-51)$$

1-8-3 Method of Percentiles: Estimators:

This method of estimation based on the aggregate density function, where we assume that $w_i$is the estimator of the aggregate distribution function$F(t_i)$, and thus we can find estimators that make the function $\sum_{i=1}^{n}[w_i - F(x_i)]^2$ at its lower end, as follows:

First: to estimate the parameter α in the equation:

$$F(t_i) = \frac{\alpha}{\alpha+1}\left(1-e^{-\lambda t^{\delta}}\right) + \frac{1}{\alpha+1}\left(1-e^{-\theta t i^{2}}\right) \quad \ldots\ldots (2-52)$$

Where ti represents ordered statistics, and the nonparametric estimator wi takes the following form:

$$wi = \frac{i}{n+1}$$

$$wi = \frac{i+\frac{3}{8}}{n+\frac{1}{4}}$$

$$wi = F(t; \alpha; \theta; \lambda; \delta)$$

When equating wi to the aggregate distribution function, we get:

$$w_i = \left[\frac{\alpha}{(1+\alpha)}\left[1-e^{-t^{\delta}\lambda}\right] + \frac{1}{1+\alpha}\left[1-e^{-\theta t i^{2}}\right]\right]$$

Taking the logarithm of both sides of the equation, we get:

$$\ln\ wi = \ln\left[\frac{\alpha}{(1+\alpha)}\left[1-e^{-t^{\delta}\lambda}\right] + \frac{1}{1+\alpha}\left[1-e^{-\theta t i^{2}}\right]\right]$$

When the equation is equal to zero, squared and added to it, we get:

$$\sum_{i=1}^{n}\left[\ln\ wi - \ln\left[\frac{\alpha}{(1+\alpha)}\left[1-e^{-t^{\delta}\lambda}\right] + \frac{1}{1+\alpha}\left[1-e^{-\theta t i^{2}}\right]\right]\right]^2 = 0 \quad \ldots\ldots (2-53)$$

By talking desecration with respect to derive the parameter α and divide by 2 as:

$$\sum_{i=1}^{n}\left[\ln(w_i)+\ln(\hat{\alpha}+1)-\ln\left(\hat{\alpha}-\hat{\alpha}e^{-\hat{\lambda}t^{\hat{\delta}}}+\left(1-e^{-\hat{\theta}t^{2}}\right)\right)\right]\left[\frac{1}{\hat{\alpha}}-\frac{1-e^{-\lambda t_i^{\hat{\delta}}}}{\left(\hat{\alpha}-\hat{\alpha}e^{-\hat{\lambda}t_i^{\hat{\delta}}}\right)+\left(1-e^{-\hat{\theta}t_i^{2}}\right)}\right] = 0 \ldots (2-54)$$

We get:

$$\frac{\partial lnPEM}{\partial \alpha}=-2\sum_{i=1}^{n}\frac{\left(-\frac{1-e^{-t^2\theta}}{(1+\alpha)^2}-\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{(1+\alpha)^2}+\frac{1-e^{-t^{\delta}\lambda}}{1+\alpha}\right)\left(\mathrm{Ln}[wi]-\mathrm{Ln}\left[\frac{1-e^{-t^2\theta}}{1+\alpha}+\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{1+\alpha}\right]\right)}{\frac{1-e^{-t^2\theta}}{1+\alpha}+\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{1+\alpha}}$$

$$=0 \quad (2-55)$$

Second: Estimation of the parameter λ

We derive equation (2-53) with respect to λ to bring the equation to its minimum and by dividing by 2 we get:

$$\frac{\partial lnPEM}{\partial \lambda}=-2\sum_{i=1}^{n}\frac{e^{-t^{\delta}\lambda}t^{\delta}\alpha\left(\mathrm{Ln}[wi]-\mathrm{Ln}\left[\frac{1-e^{-t^2\theta}}{1+\alpha}+\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{1+\alpha}\right]\right)}{(1+\alpha)\left(\frac{1-e^{-t^2\theta}}{1+\alpha}+\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{1+\alpha}\right)}=0 \qquad (2-56)$$

Third: Estimation of the parameter δ

We derive equation (2-53) with respect to δ to bring the equation to its minimum and by dividing by 2 we get:

$$\frac{\partial lnPEM}{\partial \theta}=-2\sum_{i=1}^{n}\frac{e^{-t^2\theta}t^2(\mathrm{Lnwi}]-\mathrm{Ln}[\frac{1-e^{-t^2\theta}}{1+\alpha}+\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{1+\alpha}])}{(1+\alpha)\left(\frac{1-e^{-t^2\theta}}{1+\alpha}+\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{1+\alpha}\right)}=0 \qquad (2-57)$$

Fourth: Estimation of the parameter θ

We derive equation (2-53) with respect to θ to bring the equation to its minimum and by dividing by 2 we get:

$$\frac{\partial \mathrm{lnPEM}}{\partial \delta}=-2\sum_{i=1}^{n}\frac{e^{-t^{\delta}\lambda}t^{\delta}\alpha\lambda\mathrm{Ln}[t]\left(\mathrm{Ln}[wi]-\mathrm{Ln}\left[\frac{1-e^{-t^2\theta}}{1+\alpha}+\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{1+\alpha}\right]\right)}{(1+\alpha)\left(\frac{1-e^{-t^2\theta}}{1+\alpha}+\frac{\left(1-e^{-t^{\delta}\lambda}\right)\alpha}{1+\alpha}\right)}=0 \quad (2-58)$$

By equations (2-55), (2-56), (2-57), (2-58) we get a system of nonlinear equations and using the (f.solve) method we get the estimators of the partial estimators method $(\hat{\alpha}_{per},\hat{\lambda}_{per},\hat{\delta}_{per},\hat{\theta}_{per})$ and by substituting the estimators into equation (2-22) we obtain the fuzzy reliability estimators for this method as follows:

$$\hat{R}_{per}(t)=\left(\frac{\hat{\alpha}_{per}}{\hat{\alpha}_{per}+1}\right)\left(e^{-\hat{\lambda}_{per}\tilde{A}_i t^{\hat{\delta}_{per}}}\right)+\left(\frac{1}{\hat{\alpha}_{per}+1}\right)\left(e^{-\hat{\theta}_{per}\tilde{A}_i t^2}\right) \dots (2-55)$$

1.9 Data collection stage:

Data on working times was collected from the Planning and Maintenance Department, which consists of four motors connected in parallel for a period of two years from 1/1/2020 to 12/31/2021, and all non-mechanical and non-electrical stops were excluded, and it was noted that the recorded downtimes In which there is some inaccuracy, the difference in working time occurs at a certain time, Table (1-4) shows the holiday times (in hours). while the issuance of the malfunction order is issued at a later time, and in some cases the machine may break down and then return to work on the same day before issuing the order for that malfunction, and thus results in inaccuracy in operating times Holidays lead us to the fact that holidays are hazy times.

Table (1-4) shows the breakdown times of the four engines

t_i	t_i	t_i	t_i	t_i	t_i	t_i	t_i	t_i	t_i
0.0800	0.5000	0.8000	1.2300	2.1500	3.9200	5.5800	9.0000	16.2500	32.0800
0.1200	0.5000	0.8300	1.3300	2.2300	4.2500	5.8300	9.4200	17.0000	35.3800
0.1700	0.5500	0.8300	1.3300	2.2500	4.2500	6.2700	9.5000	17.7500	35.4200
0.2000	0.5500	0.8700	1.3700	2.4200	4.2500	6.6000	10.0000	18.0000	39.4200
0.3300	0.5800	0.9500	1.5000	2.4800	4.5000	6.8300	10.2500	22.4200	40.1200
0.3300	0.6500	1.0000	1.6000	2.5800	4.5000	7.3300	10.3300	22.7500	44.5800
0.3300	0.6700	1.0000	1.8200	2.7300	4.5000	7.4200	11.0000	23.0000	50.2500
0.3500	0.7000	1.0000	1.8700	2.8200	4.5000	7.9200	11.2500	24.0000	54.7500
0.4000	0.7200	1.0500	2.0000	2.8300	4.5800	8.0000	11.5000	24.0000	61.5000
0.5000	0.7500	1.1200	2.1000	3.2300	5.5000	8.5800	13.7500	24.1200	65.5000

Table (2-4) shows the statistical indicators of the sample

Index	Value
Mean	9.4970
Median	3.5750
Mode	4.5000
Std. Deviation	14.1033
Variance	198.9040
Range	65.4200
Maximum	65.5000
Minimum	0.0800

$1-10$ Goodness of Fit:

In order to find out whether the data in Table (1-4) follow the mixed distribution (or not), the Goodness of Fit test was resorted to, according to the following statistical hypothesis:

H_0: the data are (E.F)

H_1: the data are not (E.F)

To test the aforementioned hypothesis, the Chi-Squared Statistic is calculated as follows:

$$x_c{}^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

Table (3-4) Good conformity test results

Distribution	$\chi_c{}^2$	$\chi_{table}{}^2$	P-Value	Decision
WR	2.73	5.991	0.1088	Accept H_0

From the above table, the value of $\chi_c{}^2 = 2.73$ appears, which is smaller than the tabular value of $\chi_{table}{}^2 = 5.991$, as well as the value of (P-value) =0.1088 greater than (0.05). According to this, we accept the null hypothesis that the data are distributed according to the mixed distribution (WR).

Table (4-4): criteria for comparison between distributions in representing real data

Distribution	Parameter estimation	AIC	AIC_c	BIC
WE	$\hat{\lambda} = 0.1$ $\hat{\gamma} = .5$	772.6388	772.7600	777.8491

RE	$\hat{\theta} = 0.01$	1.1174e+03	1.1174e+03	1.1200e+03
WR	$\hat{\alpha} = 1.5$ $\hat{\theta} = 0.03$ $\hat{\gamma} = 0.9$ $\hat{\lambda} = 0.07$	678.1987	678.6110	688.6193

From Table (4-4), we note that the mixed distribution (WR) was better than the rest of the distributions because it has the lowest value in relation to the differentiation criteria, which means preference over the single distributions to represent the study data.

Table (5-4): the fuzzy reliability of the three distributions

i	RpWR	RpW	RpR
1	0.998857004425527	0.972111984032897	0.999507796360164
2	0.998145109666002	0.965952115232088	0.998887702129039
3	0.997138413548148	0.959607381041687	0.997772524742629
4	0.996242090273219	0.956263898517149	0.996655702566893
5	0.991425438363793	0.944173227026379	0.989942861119786
6	0.990242963584984	0.944173227026379	0.988300413493404
7	0.989460978786643	0.944173227026379	0.986794665904149
8	0.988396580858922	0.942555196139987	0.985175612223367
9	0.985429315910566	0.938712941416515	0.979859449543661
10	0.976392964082344	0.931731423423395	0.964537414865065
11	0.975314501044686	0.931731423423395	0.961269574033926
12	0.973450322617703	0.931731423423395	0.954494327620425
13	0.968326484765926	0.928521275259895	0.943633167647269
14	0.967228617238949	0.928521275259895	0.939375926392717
14	0.963752403649898	0.926670030386522	0.929572153600909
16	0.953836824159027	0.922541813799283	0.910701864159165
17	0.950414885808823	0.921406909384840	0.902347229949978
18	0.946340963647223	0.919738394983966	0.893230607699711
19	0.942225992622353	0.918647486736890	0.880826025890972
20	0.937424177169906	0.917041510179908	0.869255275455128
21	0.929420471573715	0.914440643607217	0.849021942987563
22	0.923206119662888	0.912922455443161	0.833247334101047
23	0.921861971242358	0.912922455443161	0.830469733843957
24	0.914963292450316	0.910944058292940	0.815243854138536
25	0.901915199301379	0.907131420595186	0.773178070352780

Table (6-4): Values of fuzzy reliability, density function, and aggregate function after arranging real data for working times (in ascending order)

i	t_i	Ř(x)	F	f
1	0.08	0.99148771310879	0.00851228689120565	0.123906555202205
2	0.12	0.986084480216123	0.0139155197838766	0.146385864108775
3	0.17	0.978053769554903	0.0219462304450972	0.174790678328083
4	0.20	0.972558246400586	0.0274417535994136	0.191515796681979
5	0.33	0.943237702401870	0.0567622975981299	0.257446714774037
6	0.33	0.943237702401870	0.0567622975981299	0.257446714774037
7	0.33	0.943237702401870	0.0567622975981299	0.257446714774037
8	0.35	0.937999328284631	0.0620006717153693	0.266325733654972
9	0.40	0.924161701225050	0.0758382987749503	0.286742645804934

10	0.50	0.893766717256355	0.106233282743645	0.319227305752820
11	0.50	0.893766717256355	0.106233282743645	0.319227305752820
12	0.50	0.893766717256355	0.106233282743645	0.319227305752820
13	0.55	0.877497812847813	0.122502187152187	0.331024207479570
14	0.55	0.877497812847813	0.122502187152187	0.331024207479570
15	0.58	0.867480037114217	0.132519962885783	0.336645136885060
16	0.65	0.843569523131994	0.156430476868006	0.345533094827683
17	0.67	0.836643374890744	0.163356625109256	0.347003902005640
18	0.70	0.826210580350040	0.173789419649960	0.348344404230586
19	0.72	0.819239657260978	0.180760342739022	0.348673730963146
20	0.75	0.808781924488760	0.191218075511240	0.348346662211962
21	0.80	0.791419904768502	0.208580095231498	0.345710705045152
22	0.83	0.781087899277218	0.218912100722782	0.342947794315866
23	0.83	0.781087899277218	0.218912100722782	0.342947794315866
24	0.87	0.767464480639257	0.232535519360743	0.337988958628859
25	0.95	0.740945312068304	0.259054687931696	0.324187238840891
26	1	0.725001073992310	0.274998926007690	0.313328357215906
27	1	0.725001073992310	0.274998926007690	0.313328357215906
28	1	0.725001073992310	0.274998926007690	0.313328357215906
29	1.05	0.709635528197666	0.290364471802334	0.301086366178059
30	1.12	0.689211649644746	0.310788350355254	0.282155614525165
31	1.23	0.659933954891872	0.340066045108128	0.249808810690235
32	1.33	0.636467272291964	0.363532727708036	0.219544181340211
33	1.33	0.636467272291964	0.363532727708036	0.219544181340211
34	1.37	0.627925024751473	0.372074975248527	0.207598951639044
35	1.50	0.603374373926919	0.396625626073081	0.170699480718829
36	1.60	0.587601026238713	0.412398973761287	0.145264149930808
37	1.82	0.560838640119813	0.439161359880187	0.100745978910577
38	1.87	0.556001358847506	0.443998641152494	0.0928795117717912
39	2	0.545077142656422	0.454922857343578	0.0760130062424093
40	2.10	0.537988495036239	0.462011504963761	0.0661740966395009
41	2.15	0.534783062745240	0.465216937254760	0.0621334441598920
42	2.23	0.530037086364491	0.469962913635509	0.0567178234015481
43	2.25	0.528914571056132	0.471085428943868	0.0555449674163321
44	2.42	0.520173150458123	0.479826849541877	0.0479389783609098
45	2.48	0.517354745036308	0.482645254963692	0.0460658435591001
46	2.58	0.512876543324128	0.487123456675872	0.0436243920971035
47	2.73	0.506535108078811	0.493464891921189	0.0411162353699806
48	2.82	0.502884104572588	0.497115895427412	0.0400610039569914
49	2.83	0.502484007179991	0.497515992820009	0.0399589214286673
50	3.23	0.487109128388221	0.512890871611779	0.0372513397542455
51	3.92	0.462393919523174	0.537606080476826	0.0345229953584264
52	4.25	0.451188462117831	0.548811537882169	0.0334006649600608
53	4.25	0.451188462117831	0.548811537882169	0.0334006649600608
54	4.25	0.451188462117831	0.548811537882169	0.0334006649600608
55	4.50	0.442939847709585	0.557060152290415	0.0325940677974041
56	4.50	0.442939847709585	0.557060152290415	0.0325940677974041
57	4.50	0.442939847709585	0.557060152290415	0.0325940677974041
58	4.50	0.442939847709585	0.557060152290415	0.0325940677974041
59	4.58	0.440342381936518	0.559657618063482	0.0323431357527253

60	5.50	0.411843002904636	0.588156997095364	0.0296750764096370
61	5.58	0.409477615429401	0.590522384570599	0.0294600218186396
62	5.83	0.402195158405186	0.597804841594814	0.0288034780701640
63	6.27	0.389766494164086	0.610233505835914	0.0277013552851646
64	6.60	0.380755576910971	0.619244423089029	0.0269158573120678
65	6.83	0.374625966672532	0.625374033327468	0.0263876510665145
66	7.33	0.361709366734725	0.638290633265275	0.0252898933920442
67	7.42	0.359441871269661	0.640558128730339	0.0250992344972737
68	7.92	0.347150575273849	0.652849424726151	0.0240758497099075
69	8	0.345230850619826	0.654769149380174	0.0239175064770386
70	8.58	0.331683034381353	0.668316965618647	0.0228109785864784
71	9	0.322262924089608	0.677737075910392	0.0220523994791929
72	9.42	0.313154386465771	0.686845613534229	0.0213269172417686
73	9.50	0.311453625241432	0.688546374758568	0.0211922990942210
74	10	0.301063411918909	0.698936588081091	0.0203754914721320
75	10.25	0.296018891664470	0.703981108335531	0.0199823093916455
76	10.33	0.294425263957514	0.705574736042486	0.0198585470684707
77	11	0.281458420929792	0.718541579070208	0.0188593360579449
78	11.25	0.276788333519433	0.723211666480567	0.0185027849233156
79	11.5	0.272206329570714	0.727793670429286	0.0181546210509333
80	13.75	0.234612657328350	0.765387342671650	0.0153570360263932
81	16.25	0.199488601395547	0.800511398604453	0.0128313019838416
82	17	0.190115369839424	0.809884630160576	0.0121707194454061
83	17.75	0.181222792251989	0.818777207748011	0.0115490668653783
84	18	0.178360490462358	0.821639509537642	0.0113500079097902
85	22.42	0.135080065140965	0.864919934859035	0.00840029806206337
86	22.75	0.132338300923128	0.867661699076872	0.00821720172974055
87	23	0.130301022876622	0.869698977123378	0.00808144072941467
88	24	0.122482125938494	0.877517874061506	0.00756269556851582
89	24	0.122482125938494	0.877517874061506	0.00756269556851582
90	24.12	0.121578191883466	0.878421808116535	0.00750295910276471
91	32.08	0.0749584209907653	0.925041579009235	0.00448969754534587
92	35.38	0.0615779849535915	0.938422015046409	0.00365061186582277
93	35.42	0.0614321421083429	0.938567857891657	0.00364153441516355
94	39.42	0.0485294969092998	0.951470503090700	0.00284462280026661
95	40.12	0.0465803264141867	0.953419673585813	0.00272533755317559
96	44.58	0.0359343013144058	0.964065698685594	0.00207935822890915
97	50.25	0.0259373807842445	0.974062619215756	0.00148216665059581
98	54.75	0.0200796265381739	0.979920373461826	0.00113716359241568
99	61.5	0.0137329649246661	0.986267035075334	0.000768316501519699
100	65.5	0.0109874824293578	0.989012517570642	0.000610669264630486

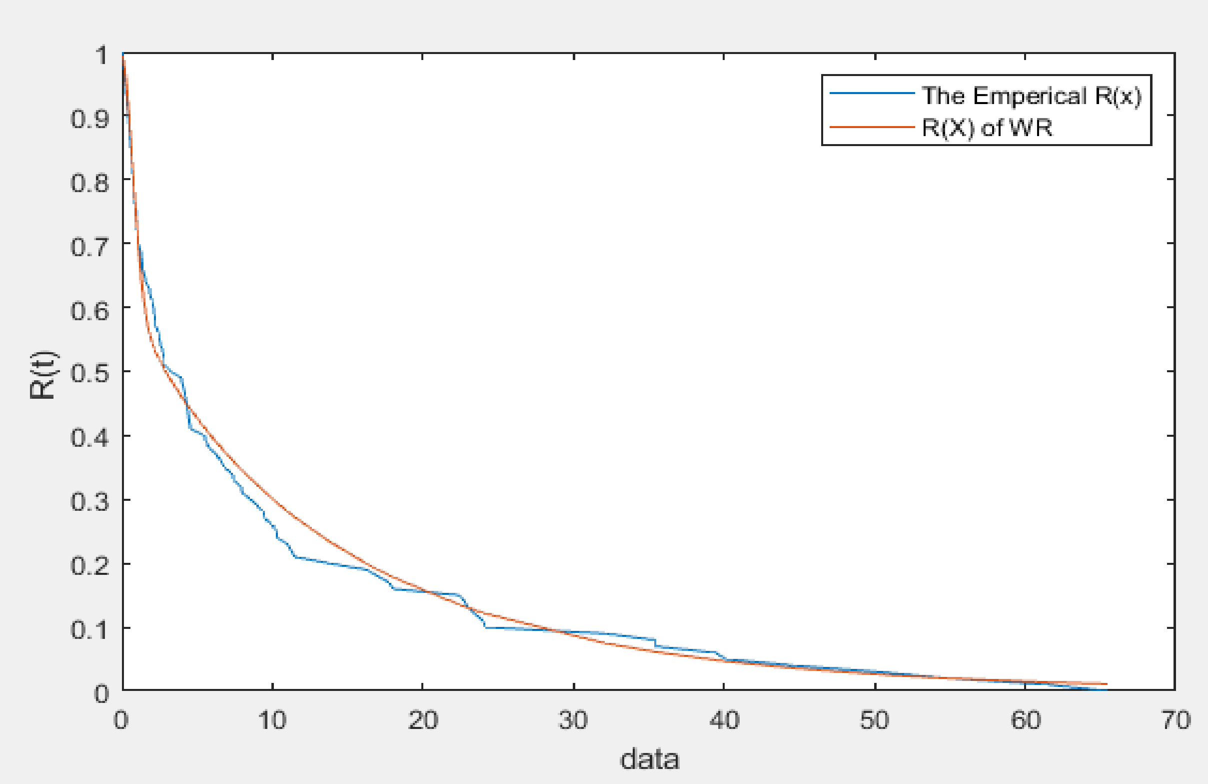


Fig (1-4): a chart showing how appropriate the three distributions of the real data are for different values of the data

1-11 Results:

The preference of the mixed distribution over the single (Whipple) and (Rayleigh) distributions in capturing the real data was evident when examining the criteria for differentiating across distributions. With time, it was found that there was less of a chance that the gadget would continue to function. The likelihood that the equipment may stop working increases with the length of time.

1-12 Recommendations:

1-The use of new types of mixed distributions due to their high efficiency and flexibility in representing operating time data.

2-Use other estimation methods (White, linear moments, Bayesian method, etc.).

3- Approving the results by the data authority (Ministry of Electricity) and using the proposed statistical distribution (Whipple-Riley) to know the reliability of electricity generation engines.

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