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Portfolio Selection Models Based On Coherent Uncertain Fuzzy Variable

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Abstract: This paper deals with uncertain portfolio selection under sensibility situations of the stock market. An investor can realize pessimistic, optimistic, or natural situations about the stock market. To acquire these conditions, a coherent uncertain fuzzy variable as an extension of the uncertain fuzzy variable is introduced. Here, the returns of the risky stocks are regarded as coherent uncertain fuzzy variables. First, we obtained coherent expected value, coherent semi-absolute deviation, and coherent skewness for coherent uncertain fuzzy variables and also reviewed some properties. Next, the coherent uncertain mean-semiabsolute deviation model and coherent uncertain mean-semiabsolute deviation-skewness model for coherent uncertain portfolio selection are presented by taking into account the bounds and cardinality constraints. To solve the proposed multi-objective optimization problem, a polynomial goal programming approach is suggested. In addition, a dominant numerical analysis of the proposed work and its comparison with existing works are presented.

Keywords: *Uncertain measure; coherent uncertain variable; coherent uncertain skewness; fuzzy portfolio selection*

1. Introduction

Portfolio selection theory concerns the optimal allocation of capital budget to risky stocks as a smart way of investment. Portfolio selection theory began with a quantitative technique [1] (Markowitz's mean-variance model), which was based on the concept that the objective of the investment is to achieve the maximum portfolio expected return with minimum portfolio risk. Markowitz's mathematical approach turned into a foundation of modern portfolio selection theory and became very famous as the mean-variance model. In Markowitz's mean-variance model returns on risky stocks were treated as random variables and the expected value of the random variable was used to describe the portfolio's expected return, and the variance of the random variable was used to quantify portfolio risk. Markowitz's model was based on the assumption that the future returns on stocks are correctly reflected from historical data. Markowitz's model was further improved and extended using probability theory. Though a large number of research works accepted probability theory as a powerful tool in portfolio selection theory, a lot of surveys and research works argued about the random nature of stock returns and the use of probability theory in portfolio selection theory because the stock market is in general unstable and is associated with ambiguities. Due to the continual occurrence of new stocks, the stock market is often affected by many non-probabilistic factors. In reality, sometimes historical returns of risky stocks have to rely on expert estimations. The fuzzy set theory [2-3] allows the inclusion of expert estimations in portfolio selection problems. The fuzzy theory was mainly developed into two categories: possibility theory and credibility theory, and portfolio selection problems were studied in both categories. With the introduction of fuzzy set theory, some scholars [4-5] described stock returns as fuzzy variables instead of random variables and studied Markowitz's mean-variance model for fuzzy portfolio selection. Some scholars [6] argued about the asymmetry of stock returns, employed the fuzzy skewness to describe the asymmetry of fuzzy return, and extended mean-variance model into MVS model. Since then numerous research works have been presented to study MVS portfolio selection models in different scenarios such as MVS model [7] with interval number as a special fuzzy number, MVS model [8] with kurtosis and semi-kurtosis, MVS model [9] considering fuzzy turnover rate as a constraint, MVS model [10] with fuzzy cross-entropy to argue about the discriminant in future return and aspiration value of return, MVS model [11] by redefining fuzzy mean and variance, MVS model [12] applying fuzzy simulation for skewness, MVS model [13] adding new burg's fuzzy entropy, MVS model [14] with trapezoidal fuzzy variable.

Although variance is widely accepted as a risk measure, it has the limitation of no distinction between gains and losses. Therefore, one of the main research directions is to explore varieties of risk measures in portfolio selection theory. Besides variance, different kinds of risk measures such as semi-variance, absolute deviation, SAD, down-side risk measure, VaR, CVaR, entropy, semi-entropy, cross-entropy, etc. were accepted in fuzzy portfolio selection theory. With semi-variance as a risk measure, numerous works were presented such as MSV model [15], MVS model [16] using the random fuzzy variable, MP model [17]

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using a trapezoidal fuzzy variable with various constraints, investor's different behavior portfolio optimization model [18] based on LR fuzzy number, MP-MSVS model [19] considering the proportion entropy, portfolio performance evaluation based MP model [20] with some realistic constraints, MO model [21] based on DEA cross efficiency, MSVS model [22] using the trapezoidal fuzzy variable.

Following the ideas of variance and semi-variance as risk measures, many author proposed absolute deviation and semi- absolute deviation, Var and CVaR, entropy and semi-entropy as risk measures in fuzzy portfolio selection theory such as, single period model [23] with absolute deviation as a risk measure, MP model [24] with absolute deviation of trapezoidal fuzzy variable as a risk measure including cardinality constraints, MP model [25] with absolute deviation as a risk measure with entropy constraints, lower absolute deviation model [26] based on varying conservative-neutral-aggressive attitudes, MO model [27] using SAD of LR-fuzzy variable as a risk measure with skewness, MP model [28] with SAD as risk measure, portfolio optimization [29] using SAD of LR power fuzzy variable to quantify portfolio risk, performance evaluation model [30] using SAD as risk measure, portfolio optimization [31] using VaR as risk measure with particle swarm optimization algorithm, MCVaR model [32] with CVaR as a new risk measure, fuzzy portfolio selection [33] based on CVaR, DEA-based portfolio efficiency evaluation models [34] with VaR and CVaR as risk measures, random fuzzy portfolio model [35] considering VaR and CVaR as risk measures, MP-MO efficient portfolio selection [36] using CVaR as risk measures, mean-entropy-skewness portfolio model [37] using entropy as a risk measure, portfolio model [38] using entropy as a risk measure with multi-choice goal programming approach to solve optimization problem, mean-semi entropy portfolio model [39] with downside risk measure via semi-entropy, MO portfolio model [40] with semi-entropy as a risk measure.

Although most of the research mentioned above works accepted the common assumption that stock returns are fuzzy variables, a paradox appears when fuzzy variables are employed to deal with subjective uncertainty. To overcome this limitation, Liu [41] proposed uncertain measures and further found uncertainty theory. Surrounding the subject, some research works on portfolio selection have been done considering stock returns are uncertain variables such as, mean-variance model [42] for uncertain portfolio optimization, concept of SAD [43] to measure downside risk was introduced, MSAD model [44] by considering the stock returns with interval expected returns as uncertain variables, mean-variance model [45] with the analytical solution, portfolio adjusting model [46] with uncertain SAD, uncertain mean-variance-skewness-kurtosis portfolio model [47] with a modified flower pollination algorithm provide to solve the optimization problem, diversified portfolio model [48] based on uncertain semi-variance with hybrid intelligent algorithm to solve the model, uncertain entropy portfolio model [49] with uncertain MO programming, mean-risk-skewness model [50], a MP uncertain portfolio problem [51] with bankruptcy constrain, uncertain portfolio problem [52] with an analytical solution, metal accounts based uncertain portfolio model [53] with realistic constraints, MP-MO uncertain portfolio model [54] with uncertain semi-entropy and skewness, optimistic value- variance-skewness uncertain model [55] with different risk preferences, risk index based uncertain model [56] with background risk, uncertain mean-variance-entropy model [57] with liquidity and diversification degree of portfolio.

According to the above-cited work stock returns can be modeled successfully by fuzzy variables or uncertain variables, but in today's era, stock markets are too complex and associated with various ambiguous factors. In practice investment decisions according to investor's different anticipations become more challenging, the uncertainty prevalent in any stock market drives the need for intelligent and flexible in-vestment strategies. In this direction, some works were presented such as those based on possibilistic measures [58] and those based on credibilistic measures [59-60]. These works are the motivation of sapidity to extend their work in uncertain theory. In this paper, we attempt to integrate the investor's attitudes mathematically in uncertain portfolio selection problems by defining the coherent uncertain variable as an extension of the uncertain variable.

The rest of the paper is organized as follows: In section 2, we provide some preliminaries about uncertain variables. In section 3, we define the coherent uncertain variable and obtain the expected value, semi-absolute deviation, and skewness of the uncertain variable, also prove and discuss some properties. In section 4, we formulate various models for portfolio selection and provide a solution methodology. In section 5, we present a numerical illustration of the proposed model and provide a discussion about the results. In section 6, we present conclusions about the proposed work.

2. Preliminaries

Uncertain variables and their measures were defined by Liu [41] who developed the uncertain theory. Furthermore, Liu [61-62] refined and extended the uncertain theory. Some definitions and properties of uncertain theory are reviewed below:

Definition 2.1. [41] Suppose that $\mathcal{L}$ be a $\sigma-$algebra over a non-empty set Γ, each element $\Lambda \in \mathcal{L}$ is called an event. The set function $\mathcal{M}\{\Lambda\}$ defined on $\mathcal{L}$ is called an uncertain measure if the following axioms hold:

Normality Axiom: $\mathcal{M}\{\Gamma\} = 1$, for the universal set Γ.

Duality Axiom: $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$, for any $\Lambda \in \mathcal{L}$.

Subadditivity Axiom: given any sequence of events Λ_1, $\Lambda_2, \Lambda_3 \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called uncertainty space. Moreover, Liu [61] defined a product uncertain measure which produces the fourth axiom as follows:

Product Axiom: Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertain spaces for $k = 1,2, \ldots, n$. Then, for the set function $\mathcal{M}$, we have

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where, Λ_k are arbitrarily chosen events from $\mathcal{L}_k$, $k = 1,2, \ldots, n$, respectively.

Theorem 2.1 [62] The uncertain measure $\mathcal{M}$ is a monotonic increasing set function, i.e. for any events $\Lambda_1, \Lambda_2 \in \mathcal{L}$ such that $\Lambda_1 \subset \Lambda_2$, $\mathcal{M}(\Lambda_1) \subset \mathcal{M}(\Lambda_2)$.

Definition 2.2 [41] An uncertain variable ξ is a function from uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that for any Borel set B of real numbers the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.

Definition 2.3 [41] The uncertainty distribution of an uncertain variable ξ is a function $\Phi: \mathcal{R} \to [0, 1]$ such that $\Phi(t) = \mathcal{M}(\xi \leq t)$ for any real number t.

An uncertainty distribution $\Phi(t)$ is said to be regular if it is a continuous and strictly increasing function with respect to t at which $0 < \Phi(t) < 1$, and

$$\lim_{t \to -\infty} \Phi(t) = 0, \quad \lim_{t \to +\infty} \Phi(t) = 1$$

Definition 2.4 [41] Let ξ be an uncertain variable, its expected value is defined as follows:

$$E(\xi) = \int_0^{\infty} \mathcal{M}(\xi \geq r)\mathrm{dr} - \int_{-\infty}^{0} \mathcal{M}(\xi \leq r)\mathrm{dr}$$

provided that at least one of the two integrals is finite.

Definition 2.5 [43] Let ξ be an uncertain variable with finite expected value $\mathrm{E(x)} = e$, them its semi-absolute deviation is defined as follows:

$$\mathrm{SAD}(\xi) = \mathrm{E}[|(\xi - e)^-|]$$

where, $(\xi - e)^- = \min(\xi - e, 0)$.

Definition 2.6 [47] Let ξ be an uncertain variable with finite expected value $\mathrm{E(x)} = e$, them its skewness is defined as follows:

$$\mathrm{SAD}(\xi) = \mathrm{E}[(\xi - \mathrm{e})^3]$$

3. The Coherent Uncertain Variables

Fuzzy portfolio selection models effectively stand out to deal with the uncertainty of the financial market. Fuzzy portfolio theory is founded upon fuzzy sets or fuzzy numbers. An uncertain variable is defined as a special case of the fuzzy set. To describe investors' different anticipations through an adaptive index k, we define the following coherent uncertain variables, which can be used to quantify stock returns with investors' different anticipations.

Definition 3.1. An uncertain variable ξ is called a coherent uncertain variable denoted by ξ_k for an adaptive index $k > 0$, if it can be characterized by a coherent uncertainty distribution which is function $\Phi_\mathrm{k}: \mathcal{R} \to [0, 1]$ defined as follows:

$$\Phi_\mathrm{k}(r) = \mathcal{M}\{\xi_k \leq r\}$$

Definition 3.2. An uncertain variable ξ_k is called coherent linear if it has a coherent uncertainty distribution

$$\Phi_\mathrm{k}(r) = \begin{cases} 0, & if \ \ r \leq a, \\ \left(\dfrac{r-a}{b-a}\right)^{\frac{1}{k}}, & if \ \ a \leq r \leq b, \\ 1, & if \ \ r \geq b. \end{cases}$$

$where\ k\ is\ an\ adaptive\ positive\ real\ number.$

The coherent uncertain linear variable is denoted by $\mathcal{L}(a, b)_k$, where a, b are real numbers with $a < b$. We write $\xi_k \sim \mathcal{L}(a, b)_k$. A coherent uncertainty distribution as an example of a typical coherent uncertain linear variable with different index k is presented geometrically (see Fig. 1).

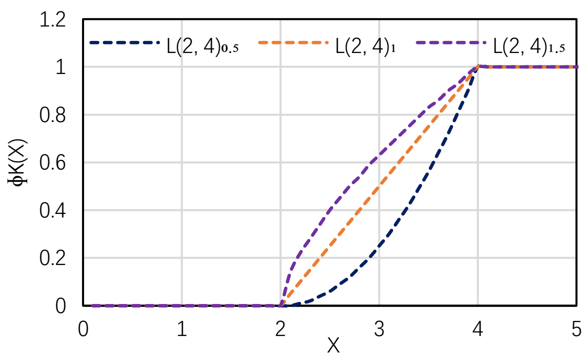

Fig. 1. The coherent uncertain distribution of coherent uncertain linear variable $\mathcal{L}(a,b)_k$ for k = 0.5, 1, 1.5

Definition 3.3. An uncertain variable ξ_k is called a coherent zigzag if it has a coherent uncertainty distribution

$$\Phi_k(r) = \begin{cases} 0, & if\ \ r \le a, \\ \left(\frac{r-a}{b-a}\right)^{\frac{1}{k}}, & if\ \ a \le r \le b, \\ 1, & if\ \ r \ge b. \end{cases}$$

where k is an adaptive positive real number.

The coherent uncertain zigzag variable is denoted by *ƶ(a, b, c)ₖ*, where a, b, c are real numbers with $a < b < c$. We write *ƶ(a, b, c)ₖ*. A coherent uncertainty distribution as an example for a typical coherent uncertain zigzag variable with different index k is presented geometrically (see Fig. 2).

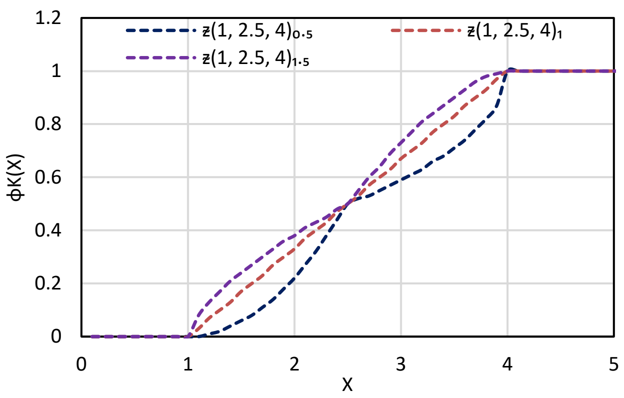

Fig. 2. The coherent uncertain distribution of coherent uncertain zigzag variable ƶ(1, 2.5, 4)$_k$ for k = 0.5, 1, 1.5

Remark 1: Investors can assume that stock returns are modeled either through coherent uncertain linear variables or through coherent uncertain zigzag variables. From Figures 1 and 2, we observed that for $k > 1$, a higher uncertainty will be associated, and the pessimism of investors will be substantiated. On the other hand for $k < 1$, a lower uncertainty will be associated and investors will be optimistic. For the case $k = 1$, investors will be neutral. This case will be the same as the uncertainty distribution for linear uncertain variables or zigzag uncertainty variables.

Remark 2: Let $\xi_{1,k} \sim \mathcal{L}(a_1, b_1)_k, \xi_{2,k} \sim \mathcal{L}(a_2, b_2)_k$ be two coherent uncertain linear variables. Then, for $\lambda > 0$, addition and scalar multiplication can be defined as follows:

$$\xi_{1,k} + \xi_{2,k} \sim \mathcal{L}(a_1 + a_2, b_1 + b_2)_k \ \ and \ \ \lambda\xi_{1,k} \sim \mathcal{L}(\lambda a_1, \lambda b_1)_k$$

which are also coherent uncertain linear variables. Similarly, we can define the addition and scalar multiplication for the coherent uncertain zigzag variable.

3.1. Expected Value of Coherent Uncertain Variable

Theorem 3.1. Let $\xi_k \sim \mathcal{L}(a,b)_k$ be a coherent uncertain linear variable. Then its expected value is given by:

$$E_k(\xi_k) = \frac{ka + b}{k + 1}$$

Theorem 3.2. Let $\xi_k \sim$ *ƶ(a, b, c)ₖ* be a coherent uncertain zigzag variable. Then its expected value is given by:

$$E_k(\xi_k) = \frac{ka + (k+1)b + c}{2(k+1)}$$

Remark 3: For an investor's extremely optimistic situation, i.e. letting $k \to 0$, we have for the coherent uncertain linear variable, the best expected possible returns: $E_k(\xi_k) = b$, and for a coherent uncertain zigzag variable: $E_k(\xi_k) = \frac{b+c}{2}$. On the other hand, for an investor's extremely pessimistic situation, i.e. letting $k \to \infty$, we have for coherent uncertain linear variable, the worst expected possible returns: $E_k(\xi_k) = a$, and for coherent uncertain zigzag variable: $E_k(\xi_k) = \frac{a+b}{2}$. Also, for the investor's neutral situation, i.e. at $k = 1$, we have the expected returns for the coherent uncertain linear variable: $E_k(\xi_k) = \frac{a+b}{2}$, which is consistent with the expected value of the linear uncertain variable as given by Liu and Qin [43] and for coherent uncertain zigzag variable: $E_k(\xi_k) = \frac{a+2b+c}{4}$, which is consistent with the expected value of the linear uncertain variable as given by Liu and Qin [43]. We can find that the interactions of coherent expected value and the underlying index k presented for four typical coherent uncertain zigzag variables (see Fig. 3).

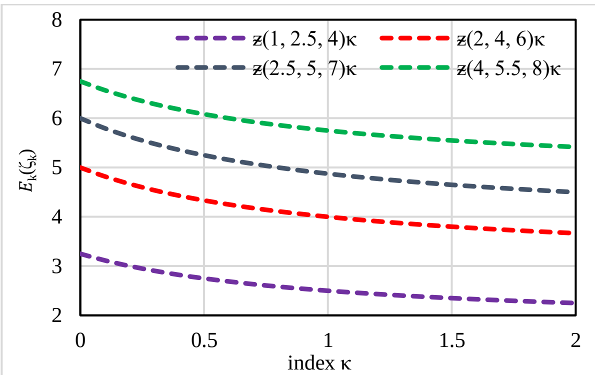

Fig. 3. The interactions of expected values $E_k(\xi_k)$ and the underlying index k for several coherent uncertain zigzag variables.

Theorem 3.3. (a) Let $\xi_{1,k}, \xi_{2,k}, \ldots, \xi_{n,k}$ are returns on n risky stocks and all are coherent uncertain linear variables. Denote $\xi_{i,k} \sim \mathcal{L}(a_i, b_i)_k$ for $i = 1, 2, 3, \ldots, n$. Also suppose that x_i be the allocation proportion to the i^{th} stock for $i = 1, 2, 3, \ldots, n$. Then, we have

$$\sum_{i=1}^{n} \xi_{i,k} x_i \sim \mathcal{L}\left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i\right)_k$$

and

$$E_k\left(\sum_{i=1}^{n}\xi_{i,k}x_i\right)=\sum_{i=1}^{n}\left(\frac{b_i+ka_i}{k+1}\right)x_i$$

(b) Let $\xi_{1,k},\xi_{2,k},\dots,\xi_{n,k}$ are returns on n risky stocks and all are coherent uncertain zigzag variables. Denote $\xi_{i,k}\sim ƶ(a_i,b_i,c_i)_k$ for $i=1,2,3,\dots,n$. Also suppose that x_i be the allocation proportion to the i^{th} stock for $i=1,2,3,\dots,n$. Then, we have

$$\sum_{i=1}^{n}\xi_{i,k}x_i\sim ƶ\left(\sum_{i=1}^{n}a_ix_i,\sum_{i=1}^{n}b_ix_i,\sum_{i=1}^{n}c_ix_i\right)_k$$

and

$$E_k\left(\sum_{i=1}^{n}\xi_{i,k}x_i\right)=\sum_{i=1}^{n}\left(\frac{ka_i+(k+1)b_i+c_i}{k+1}\right)x_i$$

3.2. Semi-Absolute Deviation of Coherent Uncertain Variable

Semi-absolute deviation is widely accepted as a risk measure for fuzzy returns. In this section, we obtain SAD of coherent linear uncertain variable and coherent zigzag uncertain variable, and review its some mathematical properties. In view of definition 2.5, the SAD of a coherent uncertain variable ξ_k with finite expected value $E_k(\xi_k)=e_k$ is defined as:

$$SAD_k(\xi_k)=E_k[|(\xi_k-e_k)^-|]$$

For simplicity, we write$(\xi_k-e_k)^-=min(\xi_k-e_k,0)$, and $(\xi_k-e_k)^+=max(\xi_k-e_k,0)$

Since $|(\xi_k-e_k)^-|$ is a non-negative coherent uncertain variable, we also have

$$SAD_k(\xi_k)=\int_0^\infty \mathcal{M}\{|(\xi_k-e_k)^-|\ \geq r\}dr$$
$$=\int_0^\infty \mathcal{M}\{e_k-\xi_k\ \geq r\}dr$$
$$=\int_0^\infty \mathcal{M}\{\xi_k\leq e_k-r\}dr$$
$$=\int_0^\infty \Phi_k(e_k-r)dr$$

This is useful relation to find the SAD. We just need coherent uncertainty distribution for the coherent uncertain variable e_k-r to obtain coherent uncertain SAD.

Theorem 3.4. Let $\xi_k\sim\mathcal{L}(a,b)_k$ be a coherent uncertain linear variable. Then it's SAD is given by:

$$SAD_k(\xi_k)=\frac{k}{(k+1)^{\frac{2k+1}{k}}}(b-a)$$

Theorem 3.5. Let $\xi_k\sim ƶ(a,b,c)_k$ be a coherent uncertain zigzag variable. Then its semi-absolute deviation is given by:

$$SAD_k(\xi_k)$$
$$=\begin{cases}\dfrac{k[(k+2)(b-a)+(c-b)]^{\frac{k+1}{k}}}{(2k+2)^{\frac{2k+1}{k}}(b-a)^{\frac{1}{k}}}, if\ \ (c-b)\leq k(b-a)\\ \dfrac{[(2k+1)(c-b)+k(b-a)]^{k+1}}{(2k+2)^{k+2}(c-b)^k}, if\ \ (c-b)\geq k(b-a)\end{cases}$$

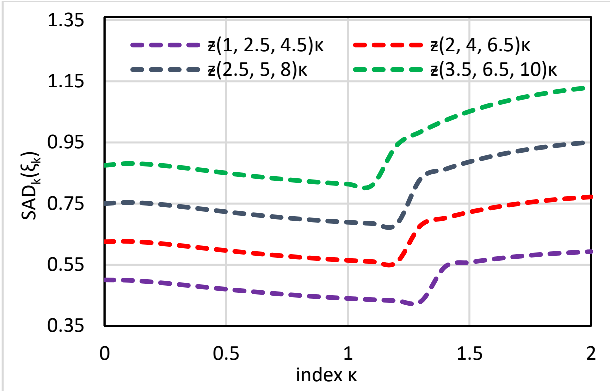

Fig. 4. The interactions of semi-absolute deviation $SAD_k(\xi_k)$ and the underlying index k for several coherent uncertain zigzag variables.

Remark 4: We note that at $k=1$, the semi-absolute deviations of a coherent uncertain linear variable $\mathcal{L}(a,b)_k$ and of a coherent uncertain zigzag variable $ƶ(a,b,c)_k$ are changed respectively as

$$SAD_k(\xi_k)=\frac{(b-a)}{8}$$

and

$$SAD_k(\xi_k)$$
$$=\begin{cases}\dfrac{[3(b-a)+(c-b)]^2}{64(b-a)}, if\ \ (c-b)\leq(b-a)\\ \dfrac{[3(c-b)+(b-a)]^2}{64(c-b)}, if\ \ (c-b)\geq(b-a)\end{cases}$$

which are consistent with semi-absolute deviations of the linear uncertain variable and uncertain zigzag variable respectively as given by Liu and Qin [43]. We can find that the interactions of coherent SAD and the underlying index k presented for four typical coherent uncertain zigzag variables (see Fig. 4), which suggests that for an investor's extremely optimistic situation, i.e. letting $k\to 0$, the coherent semi-absolute deviations are slightly increasing. On the other hand, for an investor's extremely pessimistic situation, i.e. letting $k\to\infty$, we have the worst semi-absolute deviations.

Remark 5: Let $\xi_{1,k},\xi_{2,k},\dots,\xi_{n,k}$ are returns on n risky stocks and all are coherent uncertain linear variables. Denote by $\xi_{I,k}\sim\mathcal{L}(a_I,b_i)_k$ for $i=1,2,3,\dots,n$. Also suppose that x_I be the allocation proportion to the i^{th} stock for $i=1,2,3,\dots,n$. Then, we can find the SAD of the sum $\sum_{i=1}^{n}\xi_{i,k}x_i$ as

$$SAD_k\left(\sum_{i=1}^{n}\xi_{i,k}x_i\right) = \frac{k}{(k+1)^{\frac{2k+1}{k}}}\left(\sum_{i=1}^{n}(b_i-a_i)x_i\right)$$

and, if $\xi_{1,k}, \xi_{2,k}, \dots, \xi_{n,k}$ are returns on n risky stocks and all are coherent uncertain zigzag variables $\xi_{i,k} \sim z(a_i, b_i, c_i)_k$ $for\ i = 1, 2, 3, \dots, n$. Then,

$$SAD_k\left(\sum_{i=1}^{n}\xi_{i,k}x_i\right) = \frac{k\left[(k+2)\left(\sum_{i=1}^{n}(b_i-a_i)x_i\right)+\left(\sum_{i=1}^{n}(c_i-b_i)x_i\right)\right]^{\frac{k+1}{k}}}{(2k+2)^{\frac{2k+1}{k}}\left(\sum_{i=1}^{n}(b_i-a_i)x_i\right)^{\frac{1}{k}}}$$

Or

$$SAD_k\left(\sum_{i=1}^{n}\xi_{i,k}x_i\right) = \frac{\left[(2k+1)\left(\sum_{i=1}^{n}(c_i-b_i)x_i\right)+k\left(\sum_{i=1}^{n}(b_i-a_i)x_i\right)\right]^{k+1}}{(2k+2)^{k+2}\left(\sum_{i=1}^{n}(c_i-b_i)x_i\right)^{k}}$$

3.3. Skewness of Coherent Uncertain Variable

Fuzzy returns are generally asymmetry; skewness characterizes the asymmetry of returns. In this section, we obtain skewness of coherent linear uncertain variables and coherent zigzag uncertain variables and review some mathematical properties. In view of definition 2.6, the skewness of coherent uncertain variable ξ_k with finite expected value $E_k(\xi_k) = e_k$ is defined as

$$\begin{aligned}
S_k(\xi_k) &= E_k[(\xi_k - e_k)^3] \\
&= \int_0^{\infty}\mathcal{M}\{(\xi_k - e_k)^3 \geq r\}dr - \int_{-\infty}^{0}\mathcal{M}\{(\xi_k - e_k)^3 \leq r\}dr \\
&= \int_0^{\infty}\mathcal{M}\{\xi_k - e_k \geq \sqrt[3]{r}\}dr - \int_{-\infty}^{0}\mathcal{M}\{\xi_k - e_k \leq \sqrt[3]{r}\}dr \\
&= 3\int_0^{\infty}u^2\mathcal{M}\{\xi_k - e_k \geq u\}du - 3\int_{-\infty}^{0}u^2\mathcal{M}\{\xi_k - e_k \leq u\}du \\
&= 3\int_0^{\infty}u^2[1-\Phi_k(e_k+u)]du - 3\int_{-\infty}^{0}u^2\Phi_k(e_k+u)du
\end{aligned}$$

This is useful relation to find skewness. We just need to obtain coherent uncertainty distribution $\Phi_k(e_k+u)$ for the coherent uncertain variable e_k+u to coherent uncertain skewness.

Theorem 3.6. Let $\xi_k \sim \mathcal{L}(a,b)_k$ be a coherent uncertain linear variable. Then its skewness is given by:

$$S_k(\xi_k) = \frac{2k^3(k-1)(b-a)^3}{(k+1)^3(2k+1)(3k+1)}$$

Theorem 3.7. Let $\xi_k \sim z(a, b, c)_k$ be a coherent uncertain zigzag variable. Then its skewness is given by:

$$S_k(\xi_k) = \sigma_1(b-a)^3 + \sigma_2(b-a)^2(c-b) + \sigma_3(b-a)(c-b)^2 + \sigma_4(c-b)^3$$

$$where\ \sigma_1 = \frac{3k^2}{2(k+1)^2(2k+1)} - \frac{k^3}{4(k+1)^3} - \frac{3k^3}{(k+1)(k+2)(k+3)},$$

$$\sigma_2 = \frac{3k^2}{4(k+1)^3} - \frac{3k^2}{2(k+1)^2(2k+1)},$$

$$\sigma_3 = \frac{3k}{2(k+1)^2(k+2)} - \frac{3k}{4(k+1)^3},$$

$$\sigma_4 = \frac{3}{(k+1)(k+2)(3k+1)} - \frac{3}{2(k+1)^2(k+2)} + \frac{1}{4(k+1)^3}$$

Remark 6: We note that at $k = 1$, the skewness of coherent uncertain linear variable $\mathcal{L}(a,b)_k$ and of coherent uncertain zigzag variable *z(a, b, c)$_k$* are changed respectively as

$$S_k(\xi_k) = 0$$

And

$$S_k(\xi_k) = \frac{(c-a)^2(c-2b+a)}{32}$$

which are consistent with the skewness of the linear uncertain variable and uncertain zigzag variable respectively as given in [50]. We can find that coherent skewness is decreasing w.r.t. the underlying index k from the interactions of the coherent skewness and the underlying index k for the four typical coherent uncertain zigzag variables (see Fig. 5). For investor's extremely optimistic situation, i.e. letting $k \to 0$, we have the best expected possible skewness. On the other hand, for an investor's extremely pessimistic situation, i.e. letting $k \to \infty$, we have the worst expected possible skewness.

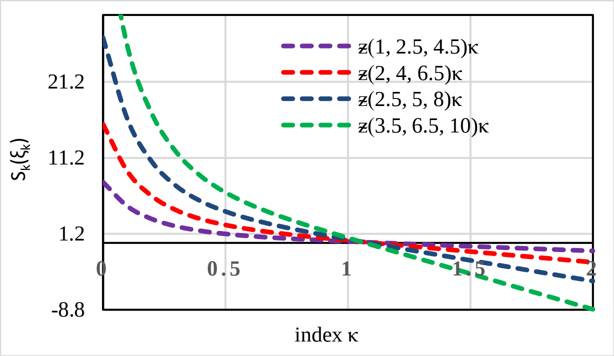

Fig. 5. The interactions of skewness $S_k(\xi_k)$ and the underlying index k for several coherent uncertain zigzag variables.

Remark 7: Let $\xi_{1,k}, \xi_{2,k}, \dots, \xi_{n,k}$ are returns on n risky stocks and all are coherent uncertain linear variables. Denote by $\xi_{i,k} \sim \mathcal{L}(a_i, b_i)_k$ for $i = 1, 2, 3, \dots, n$. Also suppose that x_i be the allocation proportion to the i^{th} stock for $i = 1, 2, 3, \dots, n$. Then, we can find the skewness of the sum $\sum_{i=1}^{n} \xi_{i,k} x_i$ as

$$S_k\left(\sum_{i=1}^{n} \xi_{i,k} x_i\right) = \frac{2k^3(k-1)}{(k+1)^3(2k+1)(3k+1)} \left(\sum_{i=1}^{n} (kb_i - a_i)x_i\right)^3$$

and, if $\xi_{1,k}, \xi_{2,k}, \dots, \xi_{n,k}$ are returns on *n* risky stocks and all are coherent uncertain zigzag variables $\xi_{i,k} \sim ƶ(a_i, b_i, c_i)_k$ *for* $i = 1, 2, 3, \dots, n$. Then,

$$S_k\left(\sum_{i=1}^{n} \xi_{i,k} x_i\right) = \sigma_1 \left(\sum_{i=1}^{n} (b_i - a_i)x_i\right)^3 + \sigma_2 \left(\sum_{i=1}^{n} (b_i - a_i)x_i\right)^2 \left(\sum_{i=1}^{n} (c_i - b_i)x_i\right) + \sigma_3 \left(\sum_{i=1}^{n} (b_i - a_i)x_i\right) \left(\sum_{i=1}^{n} (c_i - b_i)x_i\right)^2 + \sigma_4 \left(\sum_{i=1}^{n} (c_i - b_i)x_i\right)^3$$

4. Models Formation with Coherent Uncertain Variable

In this section, we describe the utility of coherent uncertain measures as described in section 3 to formulate the uncertain portfolio selection models. We formulate the CMSAD model and the coherent mean-semiabsolute deviation-skewness model using the coherent zigzag uncertain variable. Let us assume that an investor wishes to assay n risky stocks for making an intelligent investment. Also, assume that the return rates of the risky stocks are coherent uncertain zigzag variables described as $\xi_{i,k} \sim ƶ(a_i, b_i, c_i)_k$ for $i = 1, 2, 3, \dots, n$, where the adaptive index k is used to model the investor's sensibility situations about the stock market as explained in remark 1. Also suppose that x_i be the allocation proportion to the i^{th} stock for $i = 1, 2, 3, \dots, n$. Note that return on the portfolio $(x_1, x_2, \dots, x_n)$ will be the sum $\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n$, which is also a coherent uncertain zigzag variable given by

$$ƶ\left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i, \sum_{i=1}^{n} c_i x_i\right)_k$$

Assume firstly that investors are aware of the bearable risk of the portfolio and seek the portfolio return. Then, coherent expected mean of the portfolio $E_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n)$ can be maximized under the following constraints:

- The entire available budget must be invested, i.e. $\sum_{i=1}^{n} x_i = 1$.
- The maximum fraction of the entire budget imposed as u_i in a separate i^{th} stock, i.e.
$$x_i \le u_i y_i, \qquad \forall \;\; i = 1,2, \dots, n.$$
- The maximum fraction of the entire budget imposed as l_i in a separate i^{th} stock, i.e.
$$x_i \ge l_i y_i, \qquad \forall \;\; i = 1,2, \dots, n.$$
- The variable y_i takes value 1 if the i^{th} stock is included in the portfolio and 0 otherwise, i.e.
$$y_i \in \{0, 1\}, \qquad \forall \;\; i = 1,2, \dots, n.$$
- The minimum number of stocks held in the portfolio should be m out of n, i.e. $\sum_{i=1}^{n} y_i \ge m$.
- No short-selling of stocks: $0 \le x_i \le 1$, $\forall \;\; i = 1,2, \dots, n$.

Thus, taking the SAD of portfolio return as a portfolio risk the CMSAD model for portfolio selection can be described as:

$$\begin{cases} \text{Max } E_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \\ \qquad \text{subject to:} \\ \text{SAD}_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \le \pi; \; (model - 1) \\ \qquad x_1 + x_2 + \cdots + x_n = 1; \\ \qquad y_1 + y_2 + \cdots + y_n \ge m; \\ \qquad l_i y_i \le x_i \le u_i y_i \;, i = 1,2, \dots, n; \\ \qquad 0 \le x_i \le 1, i = 1,2, \dots, n; \\ \qquad \text{and} \\ \qquad y_i \in \{0\,, 1\}, i = 1,2, \dots, n. \end{cases}$$

where π is the investor's bearable risk for the portfolio selection. The objective function for maximization can be easily obtained using the coherent expected value of the coherent uncertain zigzag variable by the theorem 3.3(b). The risk function may also be easily obtained using coherent semi-absolute deviations of coherent uncertain zigzag variables as explained in remark 5. Investor's expectations can also be demonstrated by taking different indices k as explained in section 3. Similarly, assume secondly that investors are aware of the expected portfolio return and seek the portfolio risk. Then, coherent semi-absolute deviation of the portfolio return $SAD_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n)$ can be minimized under the same constraints as:

$$\begin{cases} Min\ SAD_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \\ \quad subject\ to: \\ E_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \geq \mu; \quad (model-2) \\ x_1 + x_2 + \cdots + x_n = 1; \\ y_1 + y_2 + \cdots + y_n \geq m; \\ l_i y_i \leq x_i \leq u_i y_i\ , i = 1,2,\ldots,n; \\ 0 \leq x_i \leq 1, i = 1,2,\ldots,n; \\ y_i \in \{0,1\}, i = 1,2,\ldots,n. \end{cases}$$

In general, portfolio returns are skewed. For some given specific values of expected return and risk of the portfolio, investors have a preference for higher skewness. Therefore, we present a coherent mean-semi absolute deviation-skewness model for portfolio selection. Assume firstly that investors were aware of the expected portfolio return and bearable risk of the portfolio and seek the best portfolio skewness. Then, coherent skewness of the portfolio $S_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n)$ can be maximized under the same constraints as:

$$\begin{cases} Max\ S_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \\ \quad subject\ to: \\ E_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \geq \mu; \\ SAD_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \leq \pi; \quad (model-3) \\ x_1 + x_2 + \cdots + x_n = 1; \\ y_1 + y_2 + \cdots + y_n \geq m; \\ l_i y_i \leq x_i \leq u_i y_i\ , i = 1,2,\ldots,n; \\ 0 \leq x_i \leq 1, i = 1,2,\ldots,n; \\ y_i \in \{0,1\}, i = 1,2,\ldots,n. \end{cases}$$

where μ and π are the investor's minimum return and bearable risk for the portfolio selection respectively.

Assume secondly that the investor does not aware of the expected portfolio return, risk, and portfolio skewness, we need to optimize all three objective functions simultaneously with respect to the required constraints as:

$$\begin{cases} Max\ S_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \\ Max\ E_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \\ Min\ SAD_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) \quad (model-4) \\ \quad subject\ to: \\ x_1 + x_2 + \cdots + x_n = 1; \\ y_1 + y_2 + \cdots + y_n \geq m; \\ l_i y_i \leq x_i \leq u_i y_i\ , i = 1,2,\ldots,n; \\ 0 \leq x_i \leq 1, i = 1,2,\ldots,n; \\ y_i \in \{0,1\}, i = 1,2,\ldots,n. \end{cases}$$

To solve this MO non-linear programming problem, we apply the optimal goal programming method. All the three objective functions corresponding to expected portfolio return, risk on portfolio return, and portfolio skewness can be solved separately with constraints to assign the optimal goal to the objective function. After solving these three problems and assigning optimal goals g_1, g_2, and g_3 to the objective functions the multi-objective programming problem can be reformulated to a single objective programming problem as follows

$$\begin{cases} Min\ d_1 + d_2 + d_3 \\ \quad subject\ to: \\ E_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) + d_1 = g_1; \\ SAD_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) - d_2 = g_2; \quad (mo-5) \\ S_k(\xi_{1,k}x_1 + \xi_{2,k}x_2 + \cdots + \xi_{n,k}x_n) + d_3 = g_3; \\ x_1 + x_2 + \cdots + x_n = 1; \\ y_1 + y_2 + \cdots + y_n \geq m; \\ l_i y_i \leq x_i \leq u_i y_i\ , i = 1,2,\ldots,n; \\ 0 \leq x_i \leq 1, i = 1,2,\ldots,n; \\ y_i \in \{0,1\}, i = 1,2,\ldots,n. \end{cases}$$

5. Experimental Illustration

In this section, we illustrate an experiment to verify the validity of the proposed models. We present a real-life case study using 10 stocks listed in the Bombay Stock Exchange, India (www.bseindia.com). The randomly selected 10 stocks are BAJAJ AUTO, HEG, HUBTOWN, JSW ENERGY, JSW STEEL, LUPIN, MARICO, NMDC, NTPC, ROLTA INDIA. To construct the coherent uncertain zigzag fuzzy historical returns of the stocks, we use the daily closing price of the stocks from April 1st, 2018 to April 30th, 2021 (763 observations). We divide the entire observations into three equal parts for each stock and present the average of each part as coherent uncertain zigzag fuzzy numbers for all 10 stocks with an indication of investors' expectation k. The coherent uncertain zigzag fuzzy numbers of all 10 stocks are presented in Table 1.

Table 1. The coherent uncertain zigzag fuzzy returns of the stocks with adaptive index k.

Stocks	Coherent uncertain zigzag returns
BAJAJ AUTO	$(0.154, 0.317, 0.467)_k$
HEG	$(-0.287, 0.826, 1.925)_k$
HUBTOWN	$(-0.663, -0.086, 0.467)_k$
JSW ENERGY	$(0.456, 0.628, 1.089)_k$
JSW STEEL	$(0.929, 1.489, 2.368)_k$
LUPIN	$(0.101, 0.28, 0.469)_k$
MARICO	$(0.054, 0.151, 0.303)_k$
NMDC	$(0.293, 0.513, 0.839)_k$
NTPC	$(-0.336, -0.145, 0.11)_k$
ROLTA INDIA	$(-0.676, -0.144, 0.266)_k$

Firstly, we present the computational results of the uncertain CMSAD portfolio selection model. Assume that investors are aware of the bearable portfolio risk and seek the best expected portfolio return. We solve the coherent uncertain mean-semi absolute deviation model (model 1) for portfolio selection with three different indices: k = 0.5, k = 1, k = 1.5. We solve model 1 for four different values of the investor's bearable risk level of the portfolio: 0.14, 0.16, 0.18, 0.20. We choose these risk levels carefully so that the optimal solution to the optimization problem must exist.

Table 2. The summary of the optimum solution of the coherent uncertain mean-semi absolute deviation model (model 1) with optimal stock proportions, where μ is the obtained maximized expected portfolio return under the upper bound π of the portfolio risk.

π	k	μ	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
0.14	0.5	2.2454	0.1968	0	0	0.2032	0.5	0	0	0.1	0	0
	1	2.1896	0.1	0.1	0	0.3153	0.4847	0	0	0	0	0
	1.5	2.0996	0.1	0.1	0	0.3002	0.4998	0	0	0	0	0
0.16	0.5	2.4081	0.102	0.1	0	0.298	0.5	0	0	0	0	0
	1	2.2748	0	0.1552	0	0.2448	0.5	0	0	0.1	0	0
	1.5	2.1501	0	0.1845	0	0.2155	0.5	0	0	0.1	0	0
0.18	0.5	2.5138	0	0.2083	0	0.1917	0.5	0	0	0.1	0	0
	1	2.3001	0	0.2587	0	0.1413	0.5	0	0	0.1	0	0
	1.5	2.1584	0	0.2804	0	0.1196	0.5	0	0	0.1	0	0
0.20	0.5	2.5539	0	0.2873	0	0.1127	0.5	0	0	0.1	0	0
	1	2.3102	0	0.3	0	0.1	0.5	0	0	0.1	0	0
	1.5	2.1601	0	0.3	0	0.1	0.5	0	0	0.1	0	0

To verify the impact of investors' different expectations, these risk levels remain the same for each index. Assume that the investor wants to invest whole wealth among at least 4 best stocks out of 10 stocks i.e. we take m = 4. Also assume that the investor wants to impose the lower bound as 10% and upper bound as 50% for proportions of the stocks i.e. we take $l_i = 0.1$ and $u_i = 0.5$ for i = 1, 2, …, 10. The obtained optimum computational results of model 1 are presented in Table 2. From Table 2, we observe that at the portfolio risk level 0.14, the expected portfolio returns for k = 0.5 is higher than the expected portfolio returns for k = 1, also the expected portfolio returns for k = 1 is higher than the expected portfolio returns for k = 1.5 and similar results obtained for each level of portfolio risk. We solve model 1 for more values of the portfolio risk level and obtain the efficient frontiers by the coherent uncertain expected mean and SAD with different indices k = 0.5, 1, 1.5, which are presented in Fig. 6.

We see from Fig. 6 that under the same level of portfolio risk lower expected portfolio returns obtain with larger index k, which proves the significance of the proposed model. Therefore, different expectations of the investors can be demonstrated by the incorporation of the index k in the fuzzy portfolio selection under a coherent uncertain environment. Portfolio selection model 2 for coherent uncertain mean-semi absolute deviation model is similar to model 1, investors can solve model 2 if they are aware of the expected portfolio return and seek the minimum portfolio risk. Secondly, we present the computational results of the coherent uncertain mean-semi absolute deviation-skewness model for the fuzzy portfolio selection. We use the same data set as used for the coherent uncertain mean-semi absolute deviation model. Model 3 and model 4 are very similar to model 1, hence we present the computational results of model 5 only. Assume that investors are not aware of the expected portfolio return, portfolio risk, and portfolio skewness and seek the best combination of objective function values, in that case, we solve model 5 for fuzzy portfolio selection. We solve model 5 for portfolio selection with five different indices: k = 0.4, k = 0.8, k = 1, k = 1.2, k = 1.6. Assume that the investor wants to invest whole wealth among at least 4 best stocks out of 10 stocks i.e. we take m = 4. Also assume that the investor wants to impose the lower bound as 10% and upper bound as 50% for proportions of the stocks i.e. we take l_i = 0.1 and u_i = 0.5 for i = 1, 2, …, 10. To find the optimum goals for the objective functions, we first solve individual objective functions separately subjective to all the required constraints. The obtained optimum goals for objective functions for various indices k are presented in Table 3.

We use these goals for objective functions to solve model 5. The obtained optimum computational results of Model 5 are

Table 3. the summary of optimum goals for individual objective functions in the coherent uncertain mean-semi absolute deviation-skewness model (model 5) for various indices k.

k	g_1	g_2	g_3
0.4	2.6319	0.0402	0.1576
0.8	2.3936	0.0382	0.0474
1	2.3102	0.0375	0.0138
1.2	2.242	0.037	0.0027
1.6	2.137	0.0367	-0.0002

presented in Table 4. From Table 4 we observe that all the three objective functions are decreasing as indices k are increasing. Similarly, portfolio skewness is also decreasing as indices k are increased.

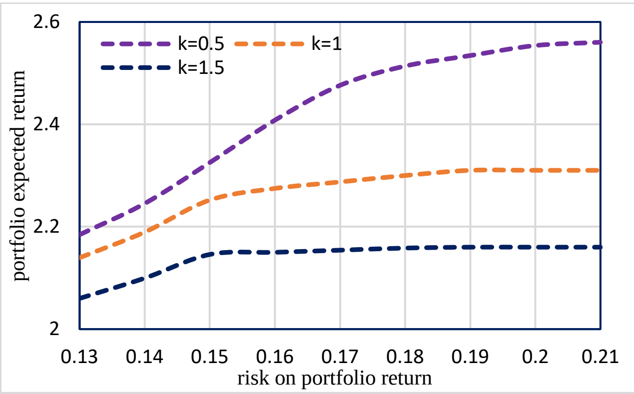

Fig. 6. The efficient frontiers of model 1 using coherent uncertain zigzag variables with different indices k which embody the expectations of the investor

We solve model 5 for more indices k and compare it with the obtained values of objective functions graphically. The intersection of portfolio expected return and underlying index k

for model 5 is presented in Fig. 7, the intersection of the portfolio risk and underlying index k is presented in Fig. 8, and the intersection of the portfolio skewness and underlying index k is presented by the Fig. 9.

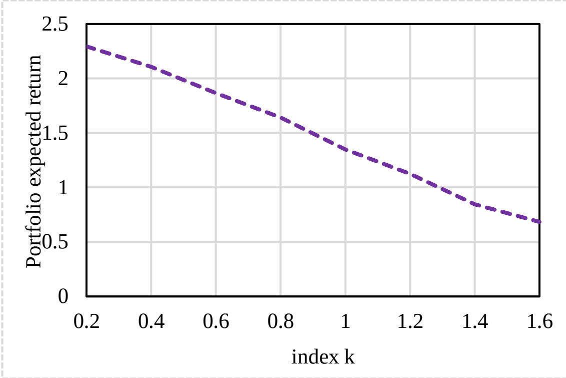

Fig. 7. The interactions of portfolio expected return and the underlying index k for coherent uncertain mean-semi absolute deviation-skewness model (model 5).

From these figures it is observed that the best combination of the objective function values can be founded for the index k less than 1.

An investor can choose any value of index k according to his/her investment objectives. Therefore, different expectations of the investors can be demonstrated by the incorporation of the index k in the fuzzy portfolio selection under a coherent uncertain environment.

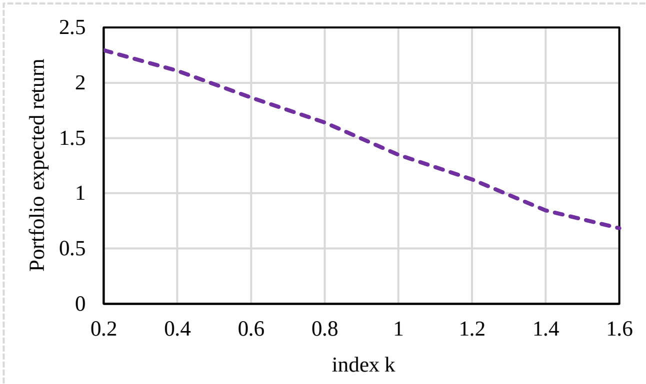

Fig. 7. The interactions of portfolio expected return and the underlying index k for coherent uncertain mean-semi absolute deviation-skewness model (model 5).

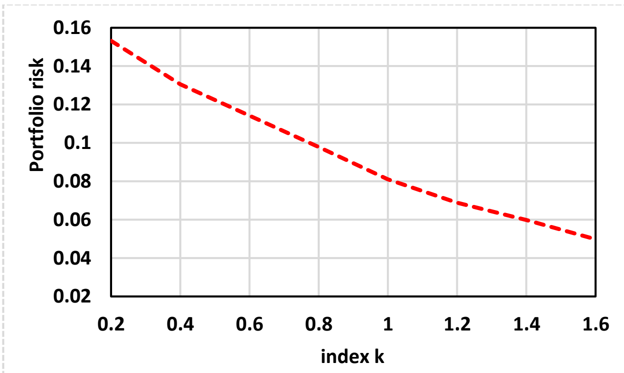

Fig. 8: The interactions of portfolio risk and the underlying index k for coherent uncertain mean-semi absolute deviation-skewness model (model 5).

Table 4: The summary of the optimum solution of the coherent uncertain mean-semi absolute deviation-skewness model (model 5) with optimal stock proportions and obtained expected portfolio return, risk, and skewness for various indices k.

k	d_1	d_2	d_3	Portfolio return	Portfolio risk	Portfolio skewness	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
0.4	0.6142	0.0903	0.1214	2.0177	0.1305	0.0362	0	0	0	0.4411	0.3589	0	0	0.1	0.1	0
0.8	0.7521	0.0616	0.04	1.6415	0.0998	0.0074	0	0	0	0.5	0.235	0	0.165	0.1	0	0
1	1.0007	0.0426	0.0112	1.3095	0.0801	0.0026	0	0	0	0.5	0.1316	0	0.2684	0.1	0	0
1.2	1.1069	0.0319	0.0018	1.1351	0.0689	0.0009	0.1	0	0	0.486	0.1	0	0.314	0	0	0
1.6	1.4529	0.0131	0	0.6841	0.0498	-0.0002	0.1	0	0	0.4045	0	0	0.3955	0	0.1	0

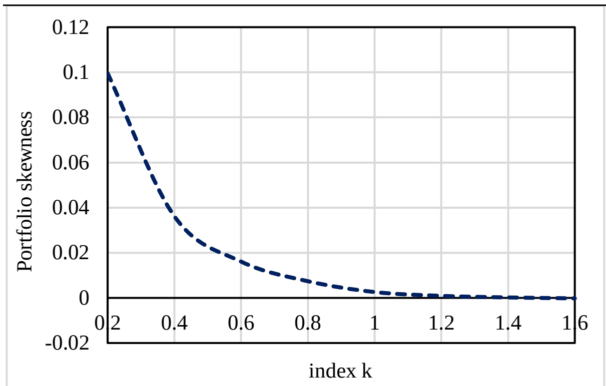

Fig. 9. The interactions of portfolio skewness and the underlying index k for coherent uncertain mean-semi absolute deviation-skewness model (model 5).

6. Conclusions

In the proposed work, we successfully coped with specific investors' expectations mathematically through the adaptive index. Also, we defined the coherent uncertain variable, and founded the coherent uncertain theory as an extension of the uncertain theory. The expected value, semi-absolute deviation, and skewness for the coherent uncertain variable were incorporated. Various properties of the expected value, semi-absolute deviation, and skewness of the coherent uncertain variable was discussed and proved. The coherent uncertain mean-semi absolute deviation model and coherent uncertain mean-semi absolute deviation-skewness model for fuzzy portfolio selection were formulated and illustrated numerically. In the case that the investors have neutral anticipations about the stock market these models reduce to the models presented by Liu and Qin [43] and Zhai et al. [50] respectively. The real data set of randomly selected 10 stocks was extracted as the coherent uncertain variable from the Indian premier market for the stock exchange to solve numerically the proposed models. Further, the obtained numerical results validate these models' application for investment in the case of sensibility situations of the stock market.

Appendix

MVS	Mean-variance-skewness
VaR	Value-at-risk
CVaR	Conditional value-at-risk
MSV	Mean- semivariance
MSVS	Mean–semivariance–skewness
DEA	Data envelopment analysis
MCVaR	Mean-conditional value-at-risk
MP	Multi-period
MO	Multi-objective
SAD	Semi-absolute deviation
MSAD	Mean-semiabsolute deviation
CMSAD	Coherent mean-semi absolute deviation

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Author Contributions

Jagdish Kumae Pahade: Conceptualization, Methodology, Writing-Original draft preparation, Software, Validation **Manoj Jha.:** Supervision, Writing-Reviewing and Editing.

Conflicts of Interest

We declare that the authors have no conflict of interest with the presented work and do not have any commercial or associative interest that represents a conflict of interest with the proposed work. All authors have read and agreed to the published version of the manuscript.

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