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INEQUALITIES FOR A CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS INVOLVING A NEW INTEGRAL OPERATOR

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Abstract. In the present paper, we investigate and introduce generalized integral operator $\mathcal{J}^{\alpha}_{m,n}$ defined on the unit disc $\mathbb{U}$. Further, we introduce subclass $\mathcal{R}^{h,p}_{\sum}(\lambda, \delta, \gamma, m, n)$ of bi-univalent function using this integral operator and obtain estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions belong to this subclass. Also connection with some results of the earlier known subclasses are mentioned.

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Key words : Analytic function, univalent function, bi-univalent function, Taylor-Maclaurin series expansion, coefficient bounds.

1. INTRODUCTION

One of the main areas of complex analysis is geometric function theory, and one of its most interesting sub-fields is the study of the univalent function theory. This subject is becoming more and more fascinating because of the variety of ideas, methods, and difficult open challenges. Therefore, the study of univalent, bi-univalent and multivalent functions continues to be an active area of current research, even after more than a century of study and research work. Because the geometric behaviour and the analytic properties of the function are interconnected, the theory of univalent functions falls neatly into the purview of the geometric function theory.

The renowned Bieberbach conjecture (1916), which de Branges ultimately resolved satisfactorily in 1985, served as the primary source of inspiration for scientists to advance the study of this topic. Researchers such as Duren [6], Goodman [7, 8],

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Nehari [14], and others have produced a variety of intriguing findings and unsolved issues related to the Bieberbach conjecture. Lewin [13] expanded on the notion of univalent functions in 1967 to include an interesting concept of bi-univalent functions. Researcher contributions to the development of the theory of bi-univalent functions include those of Netanyahu [15], Jenson and Waadeland [10], Goodman [8], Brannan and Clunie [3] (see also [4, 18]), Styer and Wright [17], Kedzierawski [11], etc.

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be an open unit disc and $G(\mathbb{U})$ be a class of all analytic functions f defined on the open unit disc $\mathbb{U}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}) \tag{1.1}$$

Let $G_{\mathbb{U}}$ be a class of all functions in $\mathbb{U}$ which are univalent in $\mathbb{U}$. The Koebe One-Quarter theorem [6] ensures that, the image of $\mathbb{U}$ under $f \in G_{\mathbb{U}}$ contains a disc of radius 1/4.

It is well known that f^{-1} exists for each $f \in G_{\mathbb{U}}$ and it is defined as

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{U}$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0, r_0(f) \geq 1/4$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function f defined on $\mathbb{U}$ is considered bi-univalent if $f \in G(\mathbb{U})$ and both f and f^{-1} are univalent on $\mathbb{U}$. $\sum$ represents the set of such bi-univalent functions defined on $\mathbb{U}$ and given by (1.1). Study of various classes of bi-univalent functions involving various operators and polynomials and also estimation on coefficient bounds of the functions belong to such classes is a fascinating field for researchers.

Lewin [13] conducted the initial investigations on this area and demonstrated that $|a_2| < 1.51$. Netanyahu [15] later demonstrated that $max|a_2| = 4/3$. Moreover, it was hypothesized by Brannan and Clunie [3] that, for $f \in \sum, |a_2| \leq \sqrt{2}$. Also several subclasses of the class bi-univalent functions $\sum$ were developed by Brannan and Taha [5]. These subclasses include the class of startlike functions $S^*(\beta)$ and the class $\mathcal{K}(\beta)$ of convex functions of order $\beta (0 \leq \beta < 1)$ in $\mathbb{U}$ (see [15]). Thereafter, on $\mathbb{U}$, the classes $S^*_{\sum}(\beta)$ of bi-starlike functions of order β and $\mathcal{K}_{\sum}(\beta)$ of bi-convex functions of order β were presented. Mathematicians first got coefficient bounds for each of these classes. Later, more congruent subclasses were developed, and numerous researchers obtained coefficient bounds for a variety of bi-univalent function subclasses (see [1, 9, 12, 16]). In this study, we estimate initial coefficient bounds for functions of the new subclass $\mathcal{R}^{h,p}_{\sum}(\lambda, \delta, \gamma, m, n)$ of $\sum$.

First we introduce an integral operator $\mathcal{J}_{m,n}$ on the class $G_{\mathbb{U}}$ of analytic functions defined as follows.

Lemma 1.1. *Let $f \in \sum$ and $m, n > 0$. The integral operator $\mathcal{J}_{m,n}$ defined as:*

$$\mathcal{J}_{m,n} : G_{\mathbb{U}} \to G_{\mathbb{U}}$$

$$\begin{aligned}\mathcal{J}_{m,n}f(z) &= \frac{1}{\beta(m+1,n+1)}\int_0^\infty \frac{t^{m-1}+t^{n-1}}{2(1+t)^{m+n+1}} f\left(\frac{tz}{1+t}\right)dt \\ &= z+\sum_{k=2}^{\infty}\mathcal{I}_{m,n}^{k}a_k z^k\end{aligned}$$

where $\mathcal{I}_{m,n}^{k}=\frac{\beta(m+k,n+1)+\beta(n+k,m+1)}{2\beta(m+1,n+1)}$ *and* $\beta(m,n)=\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt$.

Proof.

$$\begin{aligned}\mathcal{J}_{m,n}f(z) =& \frac{1}{\beta(m+1,n+1)}\int_0^\infty \frac{t^{m-1}+t^{n-1}}{2(1+t)^{m+n+1}} f\left(\frac{tz}{1+t}\right)dt \\ =& \frac{1}{\beta(m+1,n+1)}\times \\ & \int_0^\infty \left[\frac{t^{m-1}}{2(1+t)^{m+n+1}}+\frac{t^{n-1}}{1(1+t)^{m+n+1}}\right]\times\left[\frac{tz}{1+t}+\sum_{k=2}^{\infty}a_k\left(\frac{t}{1+t}\right)^k z^k\right]dt \\ =& \frac{1}{\beta(m+1,n+1)}\Bigg\{\int_0^\infty \frac{t^m z}{2(1+t)^{m+n+2}}dt+\int_0^\infty \frac{t^n z}{2(1+t)^{m+n+2}}dt+ \\ & \sum_{k=2}^{\infty}a_k z^k\left(\int_0^\infty \frac{t^{m+k-1}}{2(1+t)^{m+n+k+1}}dt+\int_0^\infty \frac{t^{n+k-1}}{2(1+t)^{m+n+k+1}}dt\right)\Bigg\} \\ =& \frac{1}{\beta(m+1,n+1)}\Bigg\{\frac{z}{2}\beta(m+1,n+1)+\frac{z}{2}\beta(n+1,m+1)+ \\ & \sum_{k=2}^{\infty}a_k z^k\left(\frac{1}{2}\beta(m+k,n+1)+\frac{1}{2}\beta(n+k,m+1)\right)\Bigg\} \\ =& z+\sum_{k=2}^{\infty}a_k\left(\frac{\beta(m+k,n+1)+\beta(n+k,m+1)}{2\beta(m+1,n+1)}\right)z^k \\ =& z+\sum_{k=2}^{\infty}a_k\mathcal{I}_{m,n}^{k}z^k.\end{aligned}$$

□

In general,

$$\mathcal{J}_{m,n}^{\alpha}f(z)=z+\sum_{k=2}^{\infty}a_k\left(\mathcal{I}_{m,n}^{k}\right)^{\alpha}z^k$$

where $\alpha\in\mathbb{N}\cup\{0\}$.

Next, we define a new class of bi-univalent functions using the integral operator $\mathcal{J}_{m,n}^{\alpha}$ as follows.

Definition 1.2. Let $f\in\mathbb{G}(U)$ be a function defined by (1.1). The class $\mathcal{R}_{\sum}^{h,p}(\lambda,\delta,\gamma,m,n)$ consists of functions $f\in\sum$ satisfying the following conditions:

(1.3)

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{\mathcal{J}_{m,n}^{\alpha}f(z)}{z}+\lambda\left(\mathcal{J}_{m,n}^{\alpha}f(z)\right)'+\delta z\left(\mathcal{J}_{m,n}^{\alpha}f(z)\right)''-1\right]\in h(U)\quad (z\in U)$$

and

(1.4)
$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}g(w)}{w}+\lambda\left(\mathcal{J}^{\alpha}_{m,n}g(w)\right)'+\delta w\left(\mathcal{J}^{\alpha}_{m,n}g(w)\right)''-1\right]\in p(U)\quad (w\in U)$$

where $\lambda\geq 1, \delta\geq 0, 0\neq\gamma\in\mathbb{C}$ and $g(w)=g^{-1}(w)$, $m,n>0$ and $\alpha\in\mathbb{N}\cup\{0\}$.

For special values of α,δ,γ and λ, the class $\mathcal{R}^{h,p}_{\sum}(\lambda,\delta,\gamma,m,n)$ reduces to many well known classes of analytic and bi-univalent functions as follows.

Remark 1.3. For $\alpha=0$ the class $\mathcal{R}^{h,p}_{\sum}(\lambda,\delta,\gamma,m,n)$ reduces to the class $S^{h,p}_{\sum}(\lambda,\delta,\gamma)$ introduced and studied by Arzu Akgil [2].

Remark 1.4. For $\alpha=0,\delta=0$ and $\gamma=1$ the class $\mathcal{R}^{h,p}_{\sum}(\lambda,\delta,\gamma,m,n)$ reduces to the class $\mathcal{B}^{h,p}_{\sum}(\lambda)$ introduced and studied by Xu et. al [19].

Remark 1.5. For $\alpha=0,\delta=0,\lambda=1$ and $\gamma=1$, the class $\mathcal{R}^{h,p}_{\sum}(\lambda,\delta,\gamma,m,n)$ reduces to the class$\mathcal{H}^{h,p}_{\sum}$ introduced and studied by Xu et. al [20].

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{R}^{h,p}_{\sum}(\lambda,\delta,\gamma,m,n)$

In this section we derive the initial coeffcient bounds of the functions from the class $\mathcal{R}^{h,p}_{\sum}(\lambda,\delta,\gamma,m,n)$.

Theorem 2.1. *Let the function $f\in\sum$ given by (1.1) belongs to the function class $\mathcal{R}^{h,p}_{\sum}(\lambda,\delta,\gamma,m,n)$. Then*

$$|a_2|\leq\min\left\{\sqrt{\frac{(|h'(0)|^2+|p'(0)|^2)|\gamma|^2}{2(1+\lambda+2\delta)^2(\mathcal{I}^2_{m,n})^{2\alpha}}},\sqrt{\frac{(|h"(0)|+|p"(0)|)|\gamma|}{2(1+2\lambda+6\delta)(\mathcal{I}^3_{m,n})^{2\alpha}}}\right\}\tag{2.1}$$

and

$$|a_3|\leq\min\left\{\frac{(|h'(0)|^2+|p'(0)|^2)|\gamma|^2}{2(1+\lambda+2\delta)^2(\mathcal{I}^2_{m,n})^{2\alpha}}+\frac{(|h"(0)|+|p"(0)|)|\gamma|}{2(1+2\lambda+6\delta)(\mathcal{I}^3_{m,n})^{\alpha}},\right.$$
$$\left.\frac{(|h''(0)|+|p''(0)|)|\gamma|(1+(\mathcal{I}^3_{m,n})^{\alpha})}{2(1+2\lambda+6\delta)(\mathcal{I}^3_{m,n})^{2\alpha})}\right\}\tag{2.2}$$

where $0\neq\gamma\in\mathbb{C},\lambda\geq 1,\delta\geq 0,z,w\in\mathbb{U}$.

Proof. Assume that, $f\in\mathcal{R}^{h,p}_{\sum}(\lambda,\delta,\gamma,m,n)$. Then from (1.3) and (1.4), we get

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}f(z)}{z}+\lambda\left(\mathcal{J}^{\alpha}_{m,n}f(z)\right)'+\delta z\left(\mathcal{J}^{\alpha}_{m,n}f(z)\right)''-1\right]=h(z)\tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}g(w)}{w}+\lambda\left(\mathcal{J}^{\alpha}_{m,n}g(w)\right)'+\delta w\left(\mathcal{J}^{\alpha}_{m,n}g(w)\right)''-1\right]=p(w),\tag{2.4}$$

where h and p are functions which satisfy the conditions of Definition 1.2. So, Taylor-Maclaurin series of p and h can be written as follow.

$$h(z)=1+h_1z+h_2z^2+\cdots$$

and

$$p(z) = 1 + p_1 w + p_2 w^2 + \cdots .$$

By using expansions of functions f and g given by (1.1) and (1.2), we have

$$\begin{aligned} &1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n} f(z)}{z} + \lambda\left(\mathcal{J}^{\alpha}_{m,n} f(z)\right)' + \delta z\left(\mathcal{J}^{\alpha}_{m,n} f(z)\right)'' - 1\right] \\ &= 1 + \frac{1}{\gamma}\sum_{k=2}^{\infty}\left[1 + (k-1)\lambda + k(k-1)\delta\right] a_k (\mathcal{I}^{k}_{m,n})^{\alpha} z^{k-1} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n} g(w)}{w} + \lambda\left(\mathcal{J}^{\alpha}_{m,n} g(w)\right)' + \delta w\left(\mathcal{J}^{\alpha}_{m,n} g(w)\right)'' - 1\right] \\ &= 1 + \frac{1}{\gamma}\left[-(1+\lambda+2\delta)(\mathcal{I}^{2}_{m,n})^{\alpha} a_2 w + (1+2\lambda+6\delta)(2a_2^2 - a_3)(\mathcal{I}^{3}_{m,n})^{\alpha} w^2 + \cdots\right] \end{aligned} \tag{2.6}$$

respectively. Next, by comparing coefficients in (2.5) and (2.6) with those of $h(z)$ and $p(w)$ respectively, we get

$$\frac{1}{\gamma}\left[1+\lambda+2\delta\right] a_2 (\mathcal{I}^{2}_{m,n})^{\alpha} = h_1, \tag{2.7}$$

$$\frac{1}{\gamma}\left[1+2\lambda+6\delta\right] a_3 (\mathcal{I}^{3}_{m,n})^{\alpha} = h_2, \tag{2.8}$$

$$-\frac{1}{\gamma}\left[1+\lambda+2\delta\right] a_2 (\mathcal{I}^{2}_{m,n})^{\alpha} = p_1, \tag{2.9}$$

and

$$\frac{1}{\gamma}\left[1+2\lambda+6\delta\right](2a_2^2 - a_3)(\mathcal{I}^{3}_{m,n})^{\alpha} = p_2. \tag{2.10}$$

From (2.7) and (2.9), we have

$$h_1 = -p_1 \tag{2.11}$$

and

$$\frac{2}{\gamma^2}(1+\lambda+2\delta)^2 a_2^2 (\mathcal{I}^{2}_{m,n})^{2\alpha} = h_1^2 + p_1^2. \tag{2.12}$$

From (2.8) and (2.10), we have

$$\frac{2}{\gamma}(1+2\lambda+6\delta) a_2^2 (\mathcal{I}^{3}_{m,n})^{2\alpha} = h_2 + p_2. \tag{2.13}$$

From (2.11) and (2.12), we have

$$|a_2|^2 \le \frac{(|h_1^2| + |p_1^2|)|\gamma|^2}{2(1+\lambda+2\delta)^2 (\mathcal{I}^{2}_{m,n})^{2\alpha}} \tag{2.14}$$

and

$$|a_2|^2 \le \frac{(|h_2| + |p_2|)|\gamma|}{2(1+2\lambda+6\delta)(\mathcal{I}^{3}_{m,n})^{2\alpha}} \tag{2.15}$$

respectively. The inequalities (2.14) and (2.15) give the desired estimate on $|a_2|$ as given in (2.1).

Further, to obtain bound on $|a_3|$, we subtract (2.10) from (2.8), which gives

$$\frac{2}{\gamma}(1+2\lambda+6\delta)(a_3-a_2^2)(\mathcal{I}_{m,n}^3)^\alpha = h_2 - p_2. \tag{2.16}$$

Now using the value of a_2^2 obtain from (2.12) in (2.16), we get

$$|a_3| \leq \frac{(|h_1|^2+|p_1|^2)|\gamma|^2}{2(1+\lambda+2\delta)^2(\mathcal{I}_{m,n}^2)^{2\alpha}} + \frac{(|h_2|+|p_2|)|\gamma|}{2(1+2\lambda+6\delta)(\mathcal{I}_{m,n}^3)^\alpha}. \tag{2.17}$$

Now using the value of a_2^2 obtain from (2.13) in (2.16), we get

$$|a_3| \leq \frac{(|h_2|+|p_2|)|\gamma|(1+(\mathcal{I}_{m,n}^3)^\alpha)}{2(1+2\lambda+6\delta)(\mathcal{I}_{m,n}^3)^{2\alpha}}. \tag{2.18}$$

Hence proved. □

3. COROLLARIES AND CONSEQUENCES

By setting particular values of α, λ, γ and δ in Theorem 2.1, we get some well known results that are mentioned below.

By setting $\alpha = 0$ in Theorem 2.1, we get the result obtained by Arzu Akgul [2].

Corollary 3.1. *If the function $f \in \sum$ given by (1.1), is in the class $S_{\sum}^{h,p}(\lambda,\delta,\gamma)$ $(0 \neq \gamma \in \mathbb{C}, \lambda \geq 1, \delta \geq 0, z, w \in \mathbb{U})$, then*

$$|a_2| \leq min\left\{\sqrt{\frac{(|h'(0)|^2+|p'(0)|^2)|\gamma|^2}{2(1+\lambda+2\delta)^2}}, \sqrt{\frac{(|h"(0)|+|p"(0)|)|\gamma|}{2(1+2\lambda+6\delta)}}\right\} \tag{3.1}$$

and

$$|a_3| \leq min\left\{\left[\frac{(|h'(0)|^2+|p'(0)|^2)|\gamma|}{2(1+\lambda+2\delta)^2} + \frac{(|h"(0)|+|p"(0)|)}{2(1+2\lambda+6\delta)}\right]|\gamma|, \frac{|h''(0)||\gamma|}{(1+2\lambda+6\delta)}\right\}. \tag{3.2}$$

By setting $\alpha = 0$ and $\gamma = 1$ in Theorem 2.1, we get the following corollary.

Corollary 3.2. *If the function $f \in \sum$ given by (1.1), is in the class $S_{\sum}^{h,p}(\lambda,\delta,\gamma)$ $(\lambda \geq 1, \delta \geq 0, z, w \in \mathbb{U})$, then*

$$|a_2| \leq min\left\{\sqrt{\frac{(|h'(0)|^2+|p'(0)|^2)}{2(1+\lambda+2\delta)^2}}, \sqrt{\frac{(|h"(0)|+|p"(0)|)}{2(1+2\lambda+6\delta)}}\right\} \tag{3.3}$$

and

$$|a_3| \leq min\left\{\frac{(|h'(0)|^2+|p'(0)|^2)|\gamma|}{2(1+\lambda+2\delta)^2} + \frac{(|h"(0)|+|p"(0)|)}{2(1+2\lambda+6\delta)}, \frac{|h''(0)|}{(1+2\lambda+6\delta)}\right\}. \tag{3.4}$$

By setting $\alpha = 0, \gamma = 1$ and $\delta = 0$ in Theorem 2.1, we get the following corollary.

Corollary 3.3. *If the function $f \in \sum$ given by (1.1), is in the class $S_{\sum}^{h,p}(\lambda,\delta,\gamma)$ $(\lambda \geq 1, z, w \in \mathbb{U})$, then*

$$|a_2| \leq min\left\{\sqrt{\frac{(|h'(0)|^2+|p'(0)|^2)}{2(1+\lambda)^2}}, \sqrt{\frac{(|h"(0)|+|p"(0)|)}{2(1+2\lambda)}}\right\} \tag{3.5}$$

and

$$|a_3| \leq min\left\{\frac{(|h'(0)|^2+|p'(0)|^2)|\gamma|}{2(1+\lambda)^2} + \frac{(|h"(0)|+|p"(0)|)}{2(1+2\lambda)}, \frac{|h''(0)|}{(1+2\lambda)}\right\}. \tag{3.6}$$

4. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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