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Powered Hazard Distribution through Order Statistics and Its Applications

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Abstract: This article addresses the distribution of order statistics of the power hazard distribution with graphical and quantitative measures along with cumulative residual entropy. We study the single and double moments and establish the recurrence relation between them. Finally, real data is analyzed to show the usefulness of our results.

Keywords: *Power hazard distribution, order statistics, moments, entropy, recurrence relations.*

1. Introduction

In probability distribution theory, the selection of a specific hazard rate for modelling real world- phenomenon depends on the flexibility of the hazard function. It is preferable to apply hazard function that best matches the set of data that is available. One leading hazard function is the power hazard function due to its acceptability over lifetime data and exhibit the flexible nature.

The power hazard (PH) distribution was proposed by Mugdadi [1] in the context of power hazard function. Due to its popularity of PH function, PH distribution has been paid more attention in the literature. Several authors have begun to work on PH distribution in classical and Bayesian statistics see Mugdadi [1] and Mugdadi and Min [2]. They regarded as the genesis for this novel distribution. Later, Ismail [3] discussed the parameter of stress-strength reliability of distribution. Khan [4-5] presented ordered random variables for PH distribution in detail.

The generalization of classical distributions remains an issue in the distribution theory. Several generalizations of PH distribution viz. length, weighted, transmuted, and extended (Mustafa and Khan [6], Khan and Mustafa [7], Khan and Mustafa [8], Mustafa and Khan [9] were reported. The moments from doubly truncated and transmuted PH distributions were discussed by Khan [10-11]. Recently, Aljohani [12] focused PH distribution in the presence of competing risks model. The review and generalization of the PH distribution is in agreement with the growing distribution since last decades.

A random variable (r.v.) X with range of values $(0,\infty)$ is said to have PH distribution, if its probability density function (PDF_{RH}) is given by

$$f_{PH}(x) = \alpha x^{\beta} e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}, x, \alpha > 0\,, \beta > -1 \quad (1)$$

The graph for PDF_{PH} is

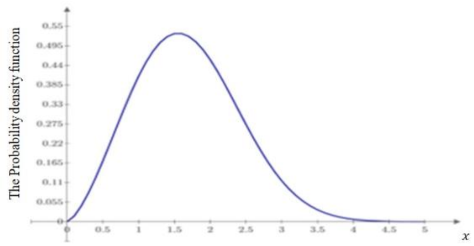

Fig. 1. The probability density function of PHD, when $\alpha = 0.5$ and $\beta = 1.5$.

The cumulative density function (CDF_{PH}) of PHdistribution is given by.

$$F_{PH}(x) = 1 - e^{-\frac{\alpha}{\beta+1}x^{\beta+1}},\ x, \alpha > 0, \beta \geq -1 \quad (2)$$

Hazard rate function

$$HRF_{PH} = \alpha x^{\beta} \quad (3)$$

HRF_{PH} of PH distribution has ability to model the failure rate of decreasing, increasing and constant of life-time model.

Reliability function

$$RF_{PH} = e^{-\frac{\alpha}{\beta+1}x^{\beta+1}} \quad (4)$$

The Rayleigh, Weibull exponential and linear exponential distributions are special case of PHdistribution at the different values of parameters.

The r^{th} moments from (1) is

$$E(X^r) = \left(\frac{\beta+1}{\alpha}\right)^{\frac{r}{\beta+1}} \Gamma\left(\frac{r}{\beta+1} + 1\right)$$

The Table 1 exhibits some statistical measures for PH

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distribution.

Table 1: The $E(X^r)$, $r = 1,2,3,4, Var(X)$ and $C.V.$ from PH distribution for different values of parameters.

	$E(X)$					$E(X^2)$				
β	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
1	1.7800	1.2540	0.8900	0.7262	0.6292	4.0000	2.0000	1.0000	0.6667	0.5000
2	1.6190	1.2816	1.0146	0.8900	0.8081	2.9717	1.8720	1.1793	0.9000	0.7425
3	1.5280	1.2831	1.0829	0.9737	0.9100	2.5172	1.7800	1.2586	1.0276	0.8900
4	1.4530	1.2696	1.104	1.0212	0.9623	2.2103	1.6751	1.2695	1.0788	0.96184
	$E(X^3)$					$E(X^4)$				
β	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
1	10.632	3.7589	1.3290	0.7234	0.4698	32.0000	8.0000	2.0000	0.8889	0.5000
2	6.0000	3.0000	1.5000	1.0000	0.7500	12.9492	5.1401	2.0398	1.1880	0.8095
3	4.3720	2.5996	1.5457	1.1404	0.9191	8.0000	4.0000	2.0000	1.3300	1.0000
4	3.5571	2.3468	1.5483	1.2139	1.0215	5.8760	3.3749	1.9383	1.4014	1.1133
	$Var(X)$					$C.V.$				
β	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
1	0.8316	0.4375	0.2079	0.1393	0.1041	0.5117	0.5271	0.5112	0.5136	0.5117
2	0.3505	0.2295	0.1499	0.1079	0.0894	0.3656	0.3737	0.3814	0.3685	0.3687
3	0.1824	0.1336	0.0859	0.0795	0.0619	0.2794	0.2844	0.2705	0.2885	0.2725
4	0.0991	0.0632	0.0506	0.0367	0.0358	0.2161	0.1977	0.2028	0.1870	0.1964

It may be concluded that from Table 1, for $\alpha \leq 1, \mathrm{E}(\mathrm{X}^{\mathrm{r}})$ is decreasing with β, while for $\alpha \geq 2$, $E(X^r)$ is increasing with β. For variability of PH distribution is decreasing with β increasing.

To our best knowledge, no order statistics of PH distribution exist. Therefore, the main aim of this article is to establish the order statistics from PH distribution, recurrence relation, entropies as well as applicability of our results by using real data sets.

2. Order Statistics (O.S.)

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from (1). Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the O.S. Then the PDF_{PH} of r^{th} O.S. $X_{r:n}$ for $1 \leq r \leq n$ is given by David and Nagaraga [13].

$$f_{PH(r:n)}(x) = C_{r:n}[F_{PH}(x)]^{r-1}[1 - F_{PH}(x)]^{n-r} f_{PH}(x), \quad (5)$$

where $x \geq 0$,

$$C_{r:n} = \frac{n!}{(r-1)!\,(n-r)!}$$

The PDF_{PH} of r^{th} O.S. is obtained as follows.

$$f_{PH(r:n)}(x) = C_{r:n}\left[1 - e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}\right]^{r-1}\left[e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}\right]^{n-r} \times \alpha x^{\beta} e^{-\frac{\alpha}{\beta+1}x^{\beta+1}} \quad (6)$$

Using binomially expansion of $\left[1 - e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}\right]^{r-1}$, we have

$$f_{PH(r:n)}(x) = C_{r:n}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i \alpha x^{\beta} e^{-\frac{(n-r+i+1)}{\beta+1}\alpha x^{\beta+1}} \quad (7)$$

The graph for PDF_{PH} of r^{th} O. S. is.

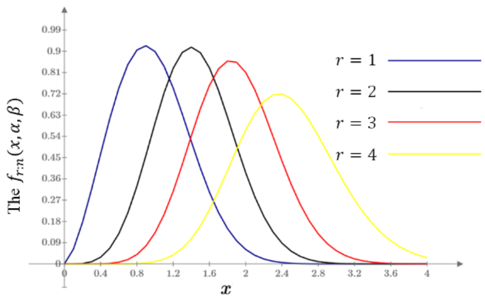


Fig. 2. Th PDF of O.S., for $n = 4, \alpha = 0.5$ and $\beta = 1.5$, for PHD.

The PDF_{PH} of r^{th} smallest O. S. is obtained at $r = 1$, as follows.

$$f_{PH(1:n)}(x) = n\alpha x^{\beta} e^{-\frac{n\alpha}{\beta+1}x^{\beta+1}} \tag{8}$$

The graph for PDF_{PH} of the smallest O. S.

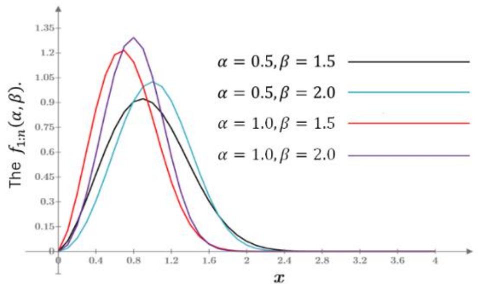


Fig. 3. The PDF of smallest O.S., when $n = 4$, for PHD.

The PDF_{PH} of r^{th} largest O. S. is obtained at $r = n$, as follows.

$$f_{PH(n:n)}(x) = n\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i \alpha x^{\beta} e^{-\frac{(i+1)}{\beta+1}\alpha x^{\beta+1}} \tag{9}$$

The graph for PDF_{PH} of the largest O. S.

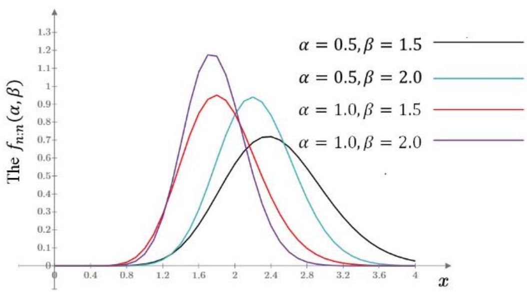


Fig. 4. The PDF of largest O.S., when $n = 4$, for PHD.

2.1 Single Moments

To infer about the underlying distribution, single moments of O. S. play an important role. It helps to evaluate the higher moments and variances of O. S. These values are presented in Tables 2-4 and 5 for the considered values of parameters.

Theorem 1: For PH distribution as reported in (1.1) and for $1 \le r \le n, k \ge 1$, we have.

$$\mu_{r:n}^{(k)} = C_{r:n}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left(\frac{\alpha}{\beta+1}\right)\times$$

$$\left[\frac{\beta+1}{\alpha(n-r+i+1)}\right]^{\frac{k+\beta+1}{\beta+1}}\Gamma\left(\frac{k}{\beta+1}+1\right)$$

Proof: We have

$$E(X_{r:n}^k) = \mu_{r:n}^{(k)} = \int_{-\infty}^{\infty} x^k f_{PH(r:n)}(x)dx\,, k = 1,2,3,\cdots \tag{10}$$

$$\mu_{r:n}^{(k)} =$$

$$C_{r:n}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\int_0^{\infty} x^{k+\beta}\alpha e^{-\frac{(n-r+i+1)}{\beta+1}\alpha x^{\beta+1}}dx. \tag{11}$$

Let $u = \frac{(n-r+i+1)}{\beta+1}\alpha x^{\beta+1}$

$$\mu_{r:n}^{(k)}$$

$$= C_{r:n}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left(\frac{\alpha}{\beta+1}\right)\left[\frac{\beta+1}{\alpha(n-r+i+1)}\right]^{\frac{k+\beta+1}{\beta+1}}$$

$$\times\int_0^{\infty} u^{\frac{k+\beta+1}{\beta+1}-1}e^{-u}du \tag{12}$$

$$\mu_{r:n}^{(k)} = C_{r:n}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left(\frac{\alpha}{\beta+1}\right)\times$$

$$\left[\frac{\beta+1}{\alpha(n-r+i+1)}\right]^{\frac{k+\beta+1}{\beta+1}}\Gamma\left(\frac{k}{\beta+1}+1\right) \tag{13}$$

Remark 1. The r^{th} moments for smallest O. S. is as follows.

$$\mu_{1:n}^{(k)} = n\left(\frac{\beta+1}{\alpha n}\right)^{\frac{k}{\beta+1}}\Gamma\left(\frac{k}{\beta+1}+1\right) \tag{14}$$

Remark 2. The r^{th} moments for largest O. S. is as follows.

$$\mu_{n:n}^{(k)} = n\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left(\frac{1}{i+1}\right)^{\frac{k}{\beta+1}+1}\times$$

$$\left(\frac{\beta+1}{\alpha}\right)^{\frac{k}{\beta+1}}\Gamma\left(\frac{k}{\beta+1}+1\right) \tag{15}$$

Theorem 1 allows us to calculate the higher moments and variances for all order statistics.

Table 2: Values of $\mu_{r:n}^{(1)} = \mu_{r:n}$, for $1 \le n \le 4, \alpha = 0.5$ and $\beta \in (0.5 - 4.0)$.

$n = 1$

β	$\mu_{1:1}$
0.5	1.87778136
1.0	1.77245385
1.5	1.68904326
2.0	1.62264973
2.5	1.56883521
3.0	1.52438119
3.5	1.48703681
4.0	1.45519939

$n = 2$

β	$\mu_{1:1}$	$\mu_{1:2}$
0.5	1.18292779	2.57263492
1.0	1.25331414	2.29159356
1.5	1.28005543	2.09803110

2.0	1.28789777	1.95740170
2.5	1.28697015	1.85070028
3.0	1.28184668	1.76691569
3.5	1.27475324	1.69932039
4.0	1.26682465	1.64357413

$n = 3$

β	$\mu_{1:3}$	$\mu_{2:3}$	$\mu_{3:3}$
0.5	0.90274529	1.74329280	2.98730597
1.0	1.02332671	1.71328899	2.58074585
1.5	1.08840937	1.66334754	2.31537287
2.0	1.12508384	1.61352561	2.12933975
2.5	1.14619048	1.56852948	1.99178568
3.0	1.15827922	1.52898158	1.88588276
3.5	1.16491630	1.49442711	1.80176702
4.0	1.16814903	1.46417588	1.73327325

$n = 4$

β	$\mu_{1:4}$	$\mu_{2:4}$	$\mu_{3:4}$	$\mu_{4:4}$
0.5	0.74519966	1.375382190	2.11120342	3.27934016
1.0	0.88622693	1.434626055	1.99195194	2.77701049
1.5	0.97010061	1.443335650	1.88335943	2.45937735
2.0	1.02220499	1.433720398	1.79333083	2.24134272
2.5	1.05574642	1.417522665	1.71953629	2.08253548
3.0	1.07790027	1.399416075	1.65854708	1.96166132
3.5	1.09277445	1.381341893	1.60751233	1.86651859
4.0	1.10283491	1.364091389	1.56426038	1.78961088

Table 3: Values of $\mu_{r:n}^{(2)}$ for $1 \leq n \leq 4, \alpha = 0.5$ and $\beta \in (0.5 - 4.0)$.

$n = 1$

β	$\mu_{1:1}^{(2)}$
0.5	5.15159106
1.0	4.000
1.5	3.37524008
2.0	2.98079294
2.5	2.70772542
3.0	2.50662827
3.5	2.35184753
4.0	2.22870594

$n = 2$

β	$\mu_{1:1}^{(2)}$	$\mu_{1:2}^{(2)}$
0.5	2.04440998	8.25877215
1.0	2.000	6.000
1.5	1.93856636	4.81191379
2.0	1.87778136	4.08380452
2.5	1.82216232	3.59328851
3.0	1.77245385	3.24080269
3.5	1.72829698	2.97539807
4.0	1.68904326	2.76836863

$n = 3$

β	$\mu_{1:3}^{(2)}$	$\mu_{2:3}^{(2)}$	$\mu_{3:3}^{(2)}$
0.5	1.19063935	3.75195124	10.51218259
1.0	1.33333333	3.33333333	7.333333333
1.5	1.40154699	3.01260509	5.711568149
2.0	1.43301923	2.767305609	4.742053982
2.5	1.44531955	2.575847863	4.102008839
3.0	1.44720251	2.422956535	3.649725780
3.5	1.44329654	2.298297862	3.313948177
4.0	1.43616477	2.194800239	3.055152822

$n = 4$

β	$\mu_{1:4}^{(2)}$	$\mu_{2:4}^{(2)}$	$\mu_{3:4}^{(2)}$	$\mu_{4:4}^{(2)}$
0.5	0.81132564	2.32858047	5.17532202	12.29113612
1.0	1.000	2.33333333	4.33333333	8.333333333
1.5	1.11341399	2.26594601	3.75926418	6.362336138
2.0	1.18292779	2.18329351	3.35131768	5.205632748
2.5	1.22622602	2.10260012	3.04909560	4.452979917
3.0	1.25331414	2.02886762	2.81704544	3.927285892
3.5	1.27006860	1.96298034	2.63361539	3.540725774
4.0	1.28005543	1.90449281	2.48510767	3.245167873

Table 4: Values of $\mu_{r:n}^{(3)}$ for $1 \leq n \leq 4, \alpha = 0.5$ and $\beta \in (0.5 - 4.0)$.

$n = 1$

β	$\mu_{1:1}^{(3)}$
0.5	18.000
1.0	10.63472311
1.5	7.600947889
2.0	6.000
2.5	5.024432117
3.0	4.371822784
3.5	3.905952713
4.0	3.557148676

$n = 2$

β	$\mu_{1:1}^{(3)}$	$\mu_{1:2}^{(3)}$
0.5	4.5	31.5
1.0	3.75994241	17.50950379
1.5	3.30850473	11.89339104
2.0	3.000	9.000
2.5	2.77370690	7.275157334
3.0	2.59950138	6.144144188
3.5	2.46059662	5.351308811
4.0	2.34684291	4.767454442

$n = 3$

β	$\mu_{1:3}^{(3)}$	$\mu_{2:3}^{(3)}$	$\mu_{3:3}^{(3)}$
0.5	2.0	9.5	42.5
1.0	2.04665342	7.18652040	22.67099549
1.5	2.03386559	5.85778301	14.91119506
2.0	2.000	5.000	11.000
2.5	1.95941174	4.40229722	8.711587391
3.0	1.91788079	3.96274257	7.234844997
3.5	1.87778136	3.62622713	6.213849651
4.0	1.84004848	3.36043177	5.470965777

$n = 4$

β	$\mu_{1:4}^{(3)}$	$\mu_{2:4}^{(3)}$	$\mu_{3:4}^{(3)}$	$\mu_{4:4}^{(3)}$
0.5	1.125	4.625	14.375	51.875
1.0	1.32934038	4.19859249	10.17444831	26.836511225446151021
1.5	1.44011032	3.81513138	7.900434643	17.24811519904080108
2.0	1.500	3.500	6.500	12.5
2.5	1.53121148	3.24401251	5.560581928	9.7619225447989281473
3.0	1.54567277	3.03450484	4.890980305	8.0161332276417378757
3.5	1.55007814	2.86089099	4.391563265	6.8212784460883450521
4.0	1.28005543	1.90449281	2.48510767	3.245167873

The validation of Equation (9) can be examined by the following expression suggested by (Arnold et al. [14]).

$$\sum_{r=1}^{n} \mu_{r:n} = nE(X)$$

Table 5: Values of $\sigma_{r:n}^2 = \mu_{r:n-}^2(\mu_{r:n})^2$, for $1 \leq n \leq 4, \alpha = 0.5$ and $\beta \in (0.5 - 4.0)$.

$n = 1$

β	$\sigma_{1:1}^2$
0.5	1.625528224
1.0	0.858407349
1.5	0.522372945
2.0	0.347800793
2.5	0.246481503
3.0	0.182890257
3.5	0.140569055
4.0	0.111100675

$n = 2$

β	$\sigma_{1:1}^2$	$\sigma_{1:2}^2$
0.5	0.645091823	1.640321718
1.0	0.429203666	0.748598955
1.5	0.300024456	0.410179293
2.0	0.219100694	0.252383104

2.5	0.165870153	0.168196983
3.0	0.129322939	0.118811634
3.5	0.103301157	0.087708282
4.0	0.084198566	0.067032709

$n = 3$

β	$\sigma^2_{1:3}$	$\sigma^2_{2:3}$	$\sigma^2_{3:3}$
0.5	0.375690291	0.712881453	1.588185632
1.0	0.286135774	0.397974166	0.67308419
1.5	0.216912033	0.245880051	0.350616621
2.0	0.167205583	0.163840714	0.207966211
2.5	0.131566933	0.115563133	0.134798644
3.0	0.105591758	0.085171863	0.093171995
3.5	0.086266553	0.064985474	0.067583782
4.0	0.071592613	0.050989231	0.050916662

$\boldsymbol{n = 4}$

$\boldsymbol{\beta}$	$\sigma^2_{1:4}$	$\sigma^2_{2:4}$	$\sigma^2_{3:4}$	$\sigma^2_{4:4}$
0.5	0.256003106	0.436904301	0.718142139	1.537064235
1.0	0.214601828	0.275181412	0.365460768	0.621546071
1.5	0.172318796	0.182728211	0.212221437	0.313799188
2.0	0.138024748	0.12773933	0.135282214	0.182015559
2.5	0.111625516	0.093229614	0.092290547	0.116025891
3.0	0.091445147	0.070502269	0.066267023	0.079170757
3.5	0.075912601	0.054874914	0.049519498	0.056834127
4.0	0.063810591	0.043747492	0.038197133	0.042460771

2. 2 Joint PDF of $X_{r:n}$ and $X_{s:n}$ O. S. for $1 \le r \le s \le n$.

The joint PDF of $X_{r:n}$ and $X_{s:n}$ is given by (Arnold et al. [14]) for $1 \le r \le s \le n$

$$f_{PH(r,s:n)}(x,y) = C_{r,s:n}[F_{PH}(x)]^{r-1}[F_{PH}(y) - F_{PH}(x)]^{s-r-1}$$
$$[1 - F_{PH}(y)]^{n-s} f_{PH}(x) f_{PH}(y), -\infty < x < y < \infty. \quad (16)$$

On using binomial expansion, (16) can be expressed as

$$f_{PH(r,s:n)}(x,y) = C_{r,s:n} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \binom{r-1}{i} \binom{s-r-1}{j} \times (-1)^{i+j}[1 - F_{PH}(x)]^{s-r+i-j-1}[1 - F_{PH}(y)]^{n-s+j} \times$$
$$f_{PH}(x) f_{PH}(y) \quad (17)$$

Therefore, the joint PDF_{PH} of $X_{r:n}$ and $X_{s:n}$ from (1) is.

$$f_{PH(r,s:n)}(x,y) = C_{r,s:n} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \binom{r-1}{i} \binom{s-r-1}{j} \times$$
$$(-1)^{i+j} \alpha^2 x^\beta e^{-(s-r+i-j-1)\frac{\alpha}{\beta+1}x^{\beta+1}} \, y^\beta e^{-(n-s+j+1)\frac{\alpha}{\beta+1}y^{\beta+1}}$$

3. Recurrence Relations

The PH distribution confirms the following relationship between the PDF_{PH} and CDF_{PH} as

$$f_{PH}(x) = \alpha x^\beta [1 - F_{PH}(x)] \quad (18)$$

Theorem 2: For $1 \le r \le n, n \in N$, we have the following relations for single moments.

$$\mu_{r:n}^{(j)} = \frac{\alpha(n-r+1)}{j+\beta+1}\left[\mu_{r:n}^{(j+\beta+1)} - \mu_{r-1:n}^{(j+\beta+1)}\right] \quad (19)$$

Proof:

$$\mu_{r:n}^{(j)} = \int_{-\infty}^{\infty} x^k f_{PH(r:n)}(x) dx$$

$$\mu_{r:n}^{(j)} = C_{r:n} \int_0^\infty x^j [F_{PH}(x)]^{r-1}[1 - F_{PH}(x)]^{n-r} f_{PH}(x) dx. \quad (20)$$

Using (18) in (20)

$$\mu_{r:n}^{(j)} = C_{r:n} \alpha \int_0^\infty x^{j+\beta} [F_{PH}(x)]^{r-1}[1 - F_{PH}(x)]^{n-r+1} dx. \quad (21)$$

Integrating by parts treating $x^{j+\beta}$ to be integrated and rest is differentiated.

$$= C_{r:n} \alpha \left(\frac{n-r+1}{j+\beta+1}\right) \times$$
$$\int_0^\infty x^{j+\beta+1} [F_{PH}(x)]^{r-1}[1 - F_{PH}(x)]^{n-r} f_{PH}(x) dx$$
$$- \frac{(r-1)}{j+\beta+1} \times$$
$$\int_0^\infty x^{j+\beta+1} [F_{PH}(x)]^{r-2}[1 - F_{PH}(x)]^{n-r+1} f_{PH}(x) dx.$$

That is,

$$\mu_{r:n}^{(j)} = \frac{\alpha(n-r+1)}{j+\beta+1}\left[\mu_{r:n}^{(j+\beta+1)} - \mu_{r-1:n}^{(j+\beta+1)}\right].$$

Theorem 3: For $1 \le r \le s \le n, n \in N$, we have the following relations for product moments.

$$\mu_{r,s:n}^{(j_1,j_2)} = \frac{\alpha(n-s+1)}{j_2+\beta+1}\left[\mu_{r,s:n}^{(j_1,j_2+\beta+1)} - \mu_{r,s-1:n}^{(j_1,j_2+\beta+1)}\right] \quad (22)$$

Proof:

$$\mu_{r,s:n}^{(j_1,j_2)} = C_{r.s:n}\int_0^\infty\int_x^\infty x^{j_1}y^{j_2}[F_{PH}(x)]^{r-1}\times$$
$$[F_{PH}(y)-F_{PH}(x)]^{s-r-1}[1-F_{PH}(y)]^{n-s}\times$$
$$f_{PH}(x)f_{PH}(y)dydx. \quad (23)$$

Or,

$$\mu_{r,s:n}^{(j_1,j_2)} = C_{r.s:n}\int_0^\infty x^{j_1}[F_{PH}(x)]^{r-1}f_{PH}(x)Z_x dx$$

where

$$Z_x = \int_x^\infty y^{j_2}[F_{PH}(y)-F_{PH}(x)]^{s-r-1}[1-F_{PH}(y)]^{n-s}\times$$
$$f_{PH}(y)dy$$

or,

$$Z_x = \alpha\int_x^\infty y^{j_2+\beta}[F_{PH}(y)-F_{PH}(x)]^{s-r-1}\times$$
$$[1-F_{PH}(y)]^{n-s+1}dy.$$

Now integrating the above equation by part,

$$Z_x = \alpha\left\{\frac{n-s+1}{j_2+\beta+1}\int_x^\infty y^{j_2+\beta+1}[F_{PH}(y)-F_{PH}(x)]^{s-r-1}\times\right.$$
$$[1-F_{PH}(y)]^{n-s}f_{PH}(y)dy-\left(\frac{s-r-1}{j_2+\beta+1}\right)\times$$
$$\int_x^\infty y^{j_2+\beta+1}[F_{PH}(y)-F_{PH}(x)]^{s-r-2}\times$$
$$\left.[1-F_{PH}(y)]^{n-s+1}f_{PH}(y)dy\right\}.$$

Putting the values of Z_x, we get the desired result.

$$\mu_{r,s:n}^{(j_1,j_2)} = \frac{\alpha(n-s+1)}{j_2+\beta+1}\left[\mu_{r,s:n}^{(j_1,j_2+\beta+1)}-\mu_{r,s-1:n}^{(j_1,j_2+\beta+1)}\right].$$

Note: Results obtained in Section 3 are valid for exponential, Weibull, Rayleigh, and linear exponential distribution as agreed by Joshi [15], Kamps [16], Lee and Kim [17] and Balakrishnan and Malik [18].

4. Entropy: To measure the average amount of uncertainty of $r.v.X$ is called entropy. There are several different versions of entropy are being used in the literature. Some noteworthy entropies are presented below.

4.1 Cumulative Residual Entropy

An alternative measure of uncertainty based on distribution function was proposed by (Rao et al. [19]) called in literature cumulative residual entropy (CRE). It extends the classical entropy (Shannon entropy). The CRE is given as

$$\psi(X) = -\int_0^\infty \bar{F}_{PH}(x)\ln\bar{F}_{PH}(x)\,dx$$

The CRE of PH distribution is.

$$\psi(X) = \frac{1}{(\beta+1)}\left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}}\Gamma\left(\frac{1}{\beta+1}+1\right)$$

Table 6: CRE of PH distribution for selected parameters**.**

α	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$	$\beta=5$
0.5	0.886	0.5408	0.3809	0.2909	0.2339
1.0	0.6264	0.4291	0.2533	0.2084	0.2084
1.5	0.5115	0.3743	0.2335	0.2335	0.1947
2.0	0.443	0.3407	0.2205	0.2205	0.1856
2.5	0.3962	0.3162	0.2108	0.2108	0.1788
3.0	0.3616	0.2976	0.2033	0.2033	0.1735
3.5	0.3348	0.2826	0.1971	0.1971	0.1691
4.0	0.3132	0.2704	0.1919	0.1919	0.1654
4.5	0.2953	0.2599	0.1874	0.1874	0.1621
5.0	0.2801	0.2509	0.1836	0.1836	0.1593

4.2 Cumulative Entropy: Crescenzo and Longobardi [20] presented the idea of cumulative entropy (CE) as follows.

$$C\psi(X) = -\int_0^\infty F_{PH}(x)\,ln\,F_{PH}\,(x)dx.$$

The CE for PH distribution is obtained as follows.

$$\ln\left(1-e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}\right) = -\sum_{n=1}^\infty\frac{e^{-\frac{\alpha}{\beta+1}nx^{\beta+1}}}{n}, x\geq\left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}}$$

$$C\psi(X) = -\int_0^\infty\left(1-e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}\right)-\sum_{n=1}^\infty\frac{e^{-\frac{\alpha}{\beta+1}nx^{\beta+1}}}{n}dx$$

$$C\psi(X) = -\int_{\left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}}}^\infty\left(1-e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}\right)-$$
$$\sum_{n=1}^\infty\frac{e^{-\frac{\alpha}{\beta+1}nx^{\beta+1}}}{n}dx.$$

After simplification, we have

$$C\psi(X) = \left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}}\left(\frac{1}{\beta+1}\right)\sum_{n=1}^\infty\frac{1}{n}\left[\left(\frac{1}{n}\right)^{\frac{1}{\beta+1}}\Gamma\left(\frac{1}{\beta+1},n\right)-\left(\frac{1}{n+1}\right)^{\frac{1}{\beta+1}}\Gamma\left(\frac{1}{\beta+1},n+1\right)\right]$$

where

$\Gamma\left(\frac{1}{\beta+1},n\right) = \int_n^\infty t^{\frac{1}{\beta+1}-1}e^{-t}dt$ is an incomplete gamma function**.**

5. Application

We analyze a real data set corresponding to survival times (in weeks) of a random sample of 33 patients suffering from Acute Myelogenous Leukemia (AML) reported by Feigl and Zelen [21] as follows.

65	156	100	134	16	108	121	4	39	143
56	26	22	1	1	5	65	56	65	17
7	16	22	3	4	2	3	8	4	3
30	4	43							

We got the MLEs of unknown parameters as: $\hat{\alpha} = 0.049$,$\hat{\beta} = -0.224$ and $LogL(\hat{\alpha}, \hat{\beta}) = -153.587$.These estimates can be utilized to know how the minimum and maximum survival times (in weeks) occur on average in every n patients. These survival times can be evaluated by $\mu_{1:n}^{(1)}$ and $\mu_{n:n}^{(1)}$, separately. From Table 7, we notice that the confirmatory analysis of $\mu_{1:n}^{(1)}$ decreases when the sample size (n) increases. On the contrary, the values of $\mu_{n:n}^{(1)}$ and variance increases as the n increases, while the corresponding skewness and kurtosis decreases when the n increases.

Table 7: The confirmatory analysis of $\mu_{1:n}^{(1)}$.

n	mean	variance	skewness	Kurtosis
5	5.11976	44.50074	2.95435	17.17008
10	2.09568	7.45619	2.95435	17.17007
15	1.24281	2.62224	2.95435	17.17008
20	0.85783	1.24930	2.95435	17.17008
25	0.64345	0.70291	2.95435	17.17008
30	0.50872	0.43936	2.95435	17.17008

Table 8: The confirmatory analysis of $\mu_{n:n}^{(1)}$.

n	Mean	variance	skewness	kurtosis
5	106.8143	5554.0	1.9045	9.3530
10	144.8468	6594.7	1.7094	8.2600
15	169.0652	7143.0	1.6317	7.8496
20	186.9981	7506.9	1.5885	7.6224
25	200.913	7923.8	1.4695	7.3970
30	213.7187	7700.5	1.7799	7.2314

The graphical illustrations based on estimated parameters, when $\hat{\alpha} = 0.049$, $\hat{\beta} = -0.224$

$PDF_{\widehat{PH}}$

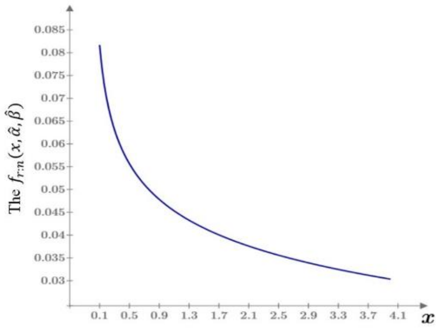


Fig. 5. Th PDF of O.S., for $n = 4$, at $\hat{\alpha}$ and $\hat{\beta}$, for PHD.

The $PDF_{\widehat{PH}}$ of smallest O. S. at $n = 4$.

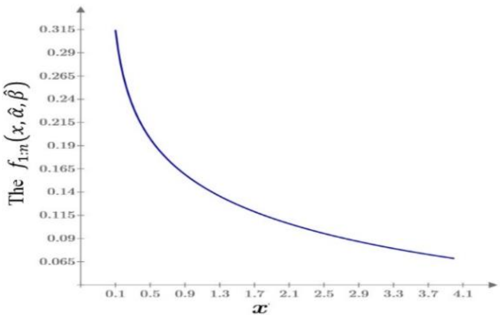


Fig. 6. Th PDF of smallest O.S., for $n = 4$, at $\hat{\alpha}$ and $\hat{\beta}$.

The $PDF_{\widehat{PH}}$ of largest O. S. at $n = 4$.

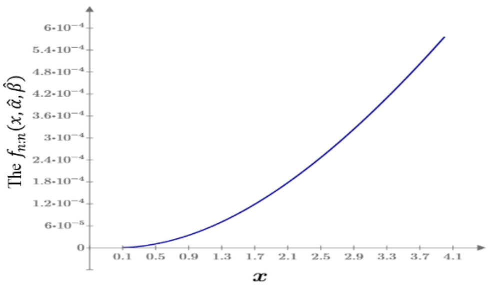
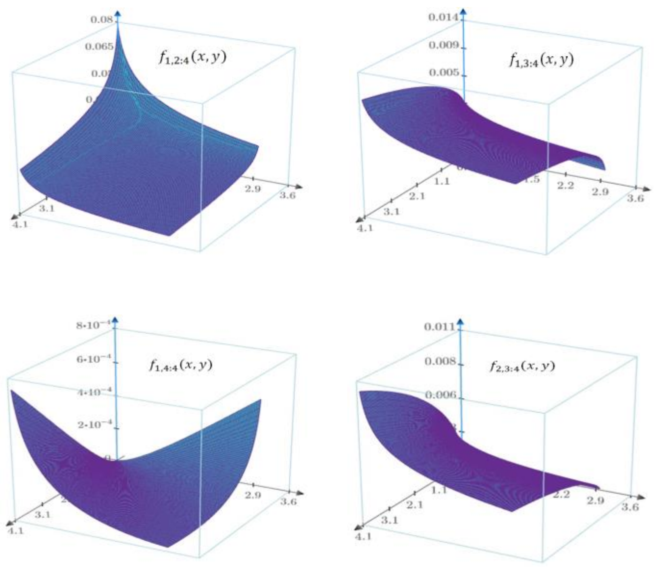


Fig. 7. Th PDF of largest O.S., for $n = 4$, at $\hat{\alpha}$ and $\hat{\beta}$.

The joint $PDF_{\widehat{PH}}$ of O. S.



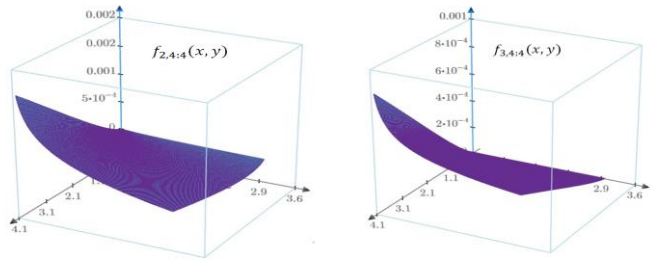


Fig. 8. Th joint PDF, $f_{r,s:n}(x,y)$ for PHD.

6. Concluding remarks

The current study presents the moments of O.S. from PH distribution. The numerical computation based on single moments of O. S. was calculated. The recursive moments for single and double moments were obtained. Finally, we fitted a medical data set using PH distribution to demonstrate the usefulness of our results.

Conflicts of interest

The authors declare no conflicts of interest.

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