

Study of Increasing Population and their Sensitivity Analysis with Second Order Differential Equations

Abha Singh

Submitted: 03/05/2024 **Revised:** 21/06/2024 **Accepted:** 28/06/2024

Abstract: Models of populations are helpful tools for investigating populations in which intraspecific interactions account for the vast bulk of the variation in the population dynamics. These models are predicated on the assumption that we can learn about a species' dynamics simply by observing it. The growth dynamics of a homogeneous population is often described using first-order differential equations (both linear and nonlinear). Second, the paper attempts to develop a second-order differential equation model of single-species population growth that is consistent with Newtonian mechanics, and third, it attempts to investigate the stochastic stability of the population system to consider environmental fluctuation in the system. In this study, we provide a second-order differential equation model of a single species' population expansion that is consistent with Newtonian physics and evolutionary theory. Clark tested the validity of this hypothesis by collecting observational data from the bighorn sheep and mule deer populations in the American West and Southwest. As a result of the stochastic stability of the deterministic model, it is more probable that the model will be right even when the random environment changes somewhat.

Keywords: Single-species, Second-order differential equation model, stochastic stability, intraspecific interaction, age distribution, population model, living organism.

1. Introduction

A population model, as the name implies, is a model that describes the population – humans, animals, bacteria, or any living organism in general – more precisely, a population model describes the changes in population size and age distribution that occur because of the population's interaction with the environment, within itself, or with other organisms. Population models based on a single species are useful tools for studying populations in which most of the dynamics is governed by intraspecific interactions. In addition, single-species models supply many of the building blocks that are utilized in multispecies models, which is very essential. These models assume that we can learn about a species' dynamics just by studying it (see [1, 2]). These models are best suited for populations that have little contact with other species. But even data on encounters, we can see flow chart of fishery model.

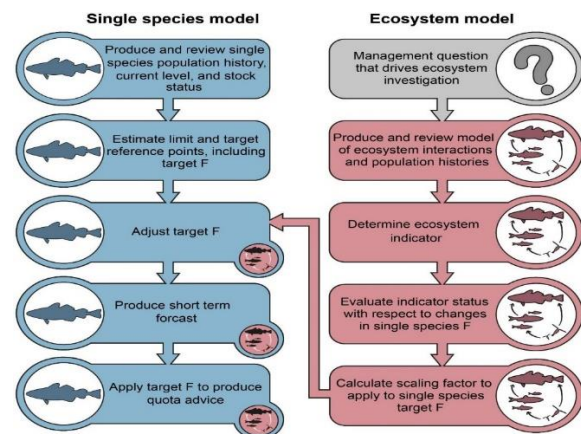


Fig 1. Flow chart outlining the steps in advice giving involved in the proposed method, with the input of the of the ecosystem modeling to the single species advice highlighted. (frontiersin.org)

The growth dynamics of a homogeneous population is usually based on first order (linear and non-linear) differential equations. It is represented as a logistics differential equation. The logistic differential equation is represented as

$$\frac{dN}{dt} = r N \left(1 - \frac{N}{K} \right)$$

It describes the situation where a population grows proportionally to its size but stops growing when it reaches the size of K (the carrying capacity) and other symbol is as r, is the linear growth rate, and N is population size. (See [1, 7, 8]). Below are showing graphical examples which are based on the logistic differential equation and show the growth of the

Department of basic science, College of Science and Theoretical Study, Dammam-Female Branch, Saudi Electronic University, Riyadh, Saudi Arabia

* Corresponding Author Email: singhabhaswdha@gmail.com, asingh@seu.edu.sa

population. In this figure represent $\alpha = 1/k$ and $N = N_0$.

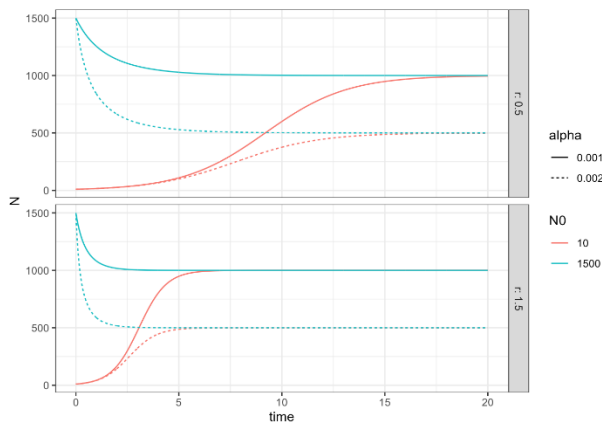


Fig 2. How variation in logistic growth parameters influence the dynamics. (See [14])

Fig 2 is showing K as an attractor because N moving toward K . After this, people have tried to figure out how to model population growth with second-order differential equations in the Newtonian Mechanics line [5, 6]. There is a controversy about the utility of second-order differential equation modeling as any higher-order differential equation can be reduced to a system of first-order differential equations [9]. However, as emphasized by Clark [5], if birth and death rates depend on historical effects along with present conditions, derivatives of higher order become interesting and realistic. For instance, we may consider a situation where the growth rate depends not on the present magnitude of the population, but on the magnitude at an earlier time not very far from the present. In such a case of short time-lag, we arrive at a second order-equation of population growth as in the case of a colony of flies [4]. Following Clark [5], here we shall consider a second-order differential equation model of a single-species population exposed to the environment. We shall extend it to the form of a stochastic differential equation to take account of the effect of the randomly fluctuation and stability of the system, along with the study of the validity of the results for some ecological systems.

2. Deterministic Model equation is used to determine the population of a single species:

Let us consider a population whose size N is increasing with time at a rate of dN/dt . In analogy with Newton's laws of motion, we assume that the population will continue to change at a particular rate until some force (for instance, scarcity of food) causes the rate to change [6]. The growth rate is a change due to either quantitative or qualitative changes in resources or in the physical environment [6]. The pattern of change in growth rates dN/dt (i.e., the second order logarithmic derivative of population size) is a basic property of species. As

emphasized by Ginzberg [6], the interpretation of ecological processes must be different depending on whether one follows the case of focusing on the acceleration rather than on the speed of a particle (system) in Newtonian Mechanics [12]. This view was defended by Ginzberg [6]. By considering a green hydra population placed in an environment with food. This results in the decline of the growth rate, and the variable of interest is the rate of change of the specific growth rate, that is, the acceleration:

$$\frac{d}{dt} \left(\frac{1}{N} \frac{dN}{dt} \right) = \frac{d^2}{dt^2} (\ln N) \quad (1)$$

So, in analogy with the Newtonian equation of motion, we assume the dynamics of population growth to be governed by the equation.

$$\frac{d^2}{dt^2} (\ln N) = F \left(\frac{d \ln N}{dt}, \ln N, P, Q \right) \quad (2)$$

Where F is the function consisting of the forces acting on the population size N or $\ln N$ and P, Q are the extrinsic and intrinsic parameters respectively. The equation (2) can be reduced to the form

$$\frac{d^2 N}{dt^2} = G(N, P, Q) \quad (3)$$

In the above modeling of equation (3) we have ignored the physiological difference through the use of averages, large-time delay, environment noises and variation in space etc. We now split the force G into two factors:

$$G = G_K + G_H \quad (4)$$

Where the first term $G_K = a(K - N)$, (5)

In equation (5) a is a constant value.

is the measure of the tendency for the population to be affected more strongly the further it is away from its equilibrium position set by the external environment? The second part is

$$G_H = -b \frac{dN}{dt} \quad (6)$$

b is a constant value in equation (6).

The force comes not only from the time it takes for environmental factors to change, but also from the resistance of the population to changes in population size that happen quickly. We have finally

$$\frac{d^2 N}{dt^2} + b \frac{dN}{dt} + a(N - K) = 0 \quad (7)$$

as the basic deterministic model equation of the population under consideration. A similar second-order differential equation for the population growth results for short time, equation (2).

This model equation was discussed by Clark [5], who also tested its validity with some observational results. Equation (7) is also our basic model equation for stochastic extension.

3. Stochastic model equation for stability analysis:

To take effect of randomly fluctuating environment of the population as we add a random force $f(t)$ to the deterministic force G leading to the stochastic differential equation.

$$\frac{d^2N}{dt^2} + b \frac{dN}{dt} + a(N - K) = f(t) \quad (8)$$

Where $f(t)$ is assumed to be a white noise characterized by

$$\langle f(t) \rangle = 0, \langle f(t)f(t') \rangle = \varepsilon \delta(t - t') \quad (9)$$

Where $\varepsilon \delta(t - t')$ is the intensity of the noise. Let us write $x = N - K$ so that $x(t)$ denotes the deviation from the equilibrium value K at any time t . Then the equation (8) reduces to the form

$$\frac{d^2x}{dt^2} + b \frac{dx}{dt} + ax = f(t) \quad (10)$$

to solve (10), let us first remove the first order derivative by setting

$$x(t) = y(t)e^{-bt/2} \quad (11)$$

Then one has the equation

$$\frac{d^2y}{dt^2} + \left(a - \frac{b^2}{4}\right)y = A(t) \quad (12)$$

$$\text{Where } A(t) = f(t) \exp(bt/2) \quad (13)$$

Let us consider the solution of the homogeneous part of the equation (12), i.e., the equation.

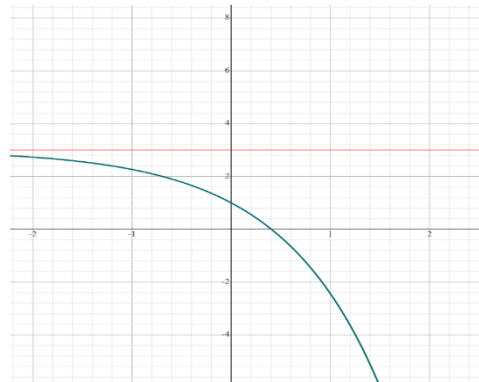
$$\frac{d^2y}{dt^2} + \left(a - \frac{b^2}{4}\right)y = 0 \quad (14)$$

Case I : Let $\Phi_1(t)$ and $\Phi_2(t)$ be the solution of (14) and are given by

$$\left. \begin{aligned} \Phi_1(t) &= e^{\left(\frac{b^2-4a}{2}\right)^{1/2} t} = e^{\lambda t} \\ \Phi_2(t) &= e^{-\left(\frac{b^2-4a}{2}\right)^{1/2} t} = e^{-\lambda t} \end{aligned} \right\} \quad (15)$$

Where

$$\lambda = \frac{(b^2 - 4a)^{1/2}}{2} > 0$$



Assuming $c_1 = -2, b = -1, a = 0, c_2 = 1, k = 2$

And we have assumed $0 \leq 4a \leq b^2$

The general solution of the equation (14) is given by

$$\begin{aligned} y &= A_1 \Phi_1(t) + A_2 \Phi_2(t) \\ &= A_1 e^{\lambda t} + A_2 e^{-\lambda t} \end{aligned}$$

Or by

$$x = N - K = A_1 e^{(\lambda-b)t} + A_2 e^{(\lambda-b)t} \quad (16)$$

$$\text{Where } \lambda = \frac{(b^2-4a)^{1/2}}{2}$$

From (16) we see that the population will remain finite for all t if

$$\begin{aligned} \lambda - b &< 0 \text{ or } b \\ &> \lambda \end{aligned} \quad (17)$$

$\Phi_1(t)$ and $\Phi_2(t)$ being given by (15) we can determine the mean-square fluctuation or variance of y satisfying the stochastic equation (12). Following Mazu [6], we have

$$\begin{aligned} \langle (\delta y)^2 \rangle &= \langle [y(t) - \langle y(t) \rangle]^2 \rangle \\ &= \int_0^t ds \int_0^t ds' G(t, s') \langle A(s)A(s') \rangle \end{aligned} \quad (18)$$

By virtue of (9), this can be written as

$$\langle (\delta y)^2 \rangle = \varepsilon \int_0^t G^2(t, s) e^{bs} ds$$

Or in terms of the original variable x , one has [6],

$$\begin{aligned} \langle (\delta y)^2 \rangle &= \varepsilon \int_0^t G^2(t, s) e^{-b(t-s)} ds \end{aligned} \quad (19)$$

$$\text{Where } G(t, s) = \Phi_1(t) \Phi_1(s) - \Phi_1(s) \Phi_1(t) \quad (20)$$

$$= e^{\lambda(t-s)} - e^{\lambda(s-t)}$$

Hence

$$\begin{aligned}
\langle (\delta N)^2 \rangle &= \langle (\delta x)^2 \rangle \\
&= \varepsilon \int_0^t [e^{2\lambda(s-t)} + e^{-2\lambda(s-t)} - 2] \\
&\quad \times e^{-bt} \cdot e^{bs} ds \\
&= \varepsilon \left[\frac{1}{b-2\lambda} \{1 - e^{-(b-2\lambda)t}\} + \frac{1}{b+2\lambda} \{1 - e^{-(b+2\lambda)t}\} \right. \\
&\quad \left. - \frac{2}{b} (1 - e^{-bt}) \right] \\
&= \varepsilon \left[\frac{1}{b-2\lambda} \{1 - e^{+(2\lambda-b)t}\} \right. \\
&\quad \left. + \frac{1}{b+2\lambda} \{1 - e^{-(2\lambda+b)t}\} \right. \\
&\quad \left. - \frac{2}{b} (1 - e^{-bt}) \right] \quad (21)
\end{aligned}$$

Which remains finite for large t if

$$b > 2\lambda \quad (22)$$

Case II: Suppose $b^2 < 4a$. Then the solution of the equation (14) is given by

$$\left. \begin{aligned} \Phi_1(t) &= \sin ut \\ \Phi_2(t) &= \cos ut \end{aligned} \right\} \quad (23)$$

Where $u = \frac{(4a-b^2)^{\frac{1}{2}}}{2}$ (24)

Then as before the mean-square fluctuation of x is given by

$$\langle (\delta x)^2 \rangle = \varepsilon \int_0^1 [\sin ut \cos us - \sin ut \cos us]^2 e^{-bt} e^{bs} ds$$

Or

$$\begin{aligned}
\langle (\delta N)^2 \rangle &= \frac{\varepsilon}{2} \left\{ \frac{1 - e^{-bt}}{b} - \frac{\cos 2ut}{b^2 + 4u^2} (b \cos 2ut \right. \\
&\quad \left. + \sin 2ut - b) \right. \\
&\quad \left. - \frac{b \sin^2 ut - \sin 4ut + 2u \sin 2ut}{b^2 + 4u^2} \right\} \quad (25)
\end{aligned}$$

Which remains finite for every value of t.

The finite value of the mean-square fluctuation $\langle (\delta N)^2 \rangle$ implies the stochastic stability of the steady state of the population system in the sense of second-order moment [9, 10].

4. Discussion and conclusion

The paper consists of two parts. In the first part (section-2) following Clark [5] and Ginzberg [6], we have presented a second-order differential equation model of a single species' population growth in line with Newtonian mechanics. Clark [5] investigated the validity of this

model using observational data from the bighorn sheep [13] and mule deer populations [11], respectively.

The second part of the paper (section 3) is concerned with the stochastic extension of the deterministic model to consider the effect of the fluctuating environment on the system. This is one of many possible extensions of the original model equation (7) suggested by Clark himself [5]. The stochastic stability, that is, the finiteness of the second-order moment (or variance) of population size for all time, implies that the real growth curve will not deviate very much from the mean or the deterministic growth curve. One point is, however, to be noted. In this case, the finiteness of the population size or equivalently the first order moment of the stochastic population is required, whereas for the stochastic stability or the finiteness of the second-order moment, it is required that. The validity of the stochastic model, therefore, requires a slightly narrow range of the parabola. The stochastic stability of the deterministic model, on the other hand, makes it more likely that the model is correct for small changes in the random environment (see [3, 12]).

Acknowledgement

The authors extend their appreciation to the Deanship of Scientific Research at Saudi Electronic University for funding this research (9482-ST5-2022-1-202301-1).

Reference

- [1] Abbott, K. & Ives, A. (2012). Single-Species Population Models. In A. Hastings & L. Gross (Ed.), *Encyclopedia of Theoretical Ecology* (pp. 641-648). Berkeley: University of California Press. <https://doi.org/10.1525/9780520951785-113>
- [2] Akçakaya, H. R., Ginzburg, L. R., Slice, D., & Slobodkin, L. B. (1988). The theory of population dynamics—II. Physiological delays. *Bulletin of Mathematical Biology*, 50(5), 503-515.
- [3] Anstead, K., Drew, K., Chagaris, D., Cieri, M., Schueller, A., McNamee, J., et al. (2020). The Past, Present, and Future of Forage Fish Management: a Case Study of Atlantic Menhaden. *Front. Mar. Sci.* Vol..
- [4] Arnold L. *Stochastic differential equations*. New York. 1974.
- [5] Banks RB. *Growth and diffusion phenomena: Mathematical frameworks and applications*. Springer Science & Business Media; 1993 Dec 22.
- [6] Bentley, J. W., Hines, D. E., Borrett, S. R., Serpetti, N., Hernandez-Milian, G., Fox, C., et al. (2019a). Combining scientific and fishers' knowledge to cocreate indicators of food web structure and function. *ICES J. Mar. Sci.* 76, 2218–2234. doi: 10.1093/icesjms/fsz121.

- [7] Bentley, J. W., Serpetti, N., Fox, C. J., Heymans, J. J., and Reid, D. G. (2020). Retrospective analysis of the influence of environmental drivers on commercial stocks and fishing opportunities in the Irish Sea. *Fisheries Oceanogr.* 29, 415–435. doi: 10.1111/fog.12486.
- [8] Bentley, J. W., Serpetti, N., Fox, C., Heymans, J. J., and Reid, D. G. (2019b). Fishers' knowledge improves the accuracy of food web model predictions. *ICES J. Mar. Sci.* 76, 897–912. doi: 10.1093/icesjms/fsz003.
- [9] Clark, J.D. the second order derivative and population modeling. *Ecology*, 52(1971), 606.
- [10] Drew, K., Cieri, M., Schueller, A., Buchheister, A., Chagaris, D., Nesslage, G., et al. (2020). Balancing Model Complexity, Data Requirements, and Management Objectives in Developing Ecological Reference Points for Atlantic Menhaden. *Front. Mar. Sci.* Vol
- [11] Ginzburg, L. R. (1987). The theory of population dynamics: back to first principles. In *Mathematical Topics in Population Biology, Morphogenesis and Neurosciences* (pp. 70-79). Springer, Berlin, Heidelberg.
- [12] Gardner, R. H., O'Neill, R. V., Mankin, J. B., & Carney, J. H. (1981). A comparison of sensitivity analysis and error analysis based on a stream ecosystem model. *Ecological Modelling*, 12(3), 173-190.
- [13] Howell, D., Schueller, A. M., Bentley, J. W., Buchheister, A., Chagaris, D., Cieri, M., ... & Townsend, H. (2021). Combining ecosystem and single-species modeling to provide ecosystem-based fisheries management advice within current management systems. *Frontiers in Marine Science*, 7, 1163.
- [14] Innis, G. (1972). The second derivative and population modeling: another view. *Ecology*, 53(4), 720-723.
- [15] ICES (2020a). Baltic Fisheries Assessment Working Group (WGBFAS). *ICES Sci. Rep.* 2:643. doi: 10.17895/ices.pub.6024
- [16] ICES (2020b). Working Group on the Assessment of Demersal Stocks in the North Sea and Skagerrak (WGNSSK). *ICES Sci. Rep.* 2:1140. doi: 10.17895/ices.pub.6092
- [17] ICES (2020c). Workshop on an Ecosystem Based Approach to Fishery Management for the Irish Sea (WKIrish6; outputs from 2019 meeting). *ICES Sci. Rep.* 2:32. doi: 10.17895/ices.pub.5551
- [18] Mazo, R. M. (1981). On the Brownian motion of a frequency modulated oscillator. *Journal of Statistical Physics*, 24(1), 39-44.
- [19] Mohler, L. L., Wampole, J. H., & Fichter, E. (1951). Mule deer in Nebraska National Forest. *The Journal of Wildlife Management*, 15(2), 129-157.
- [20] Rosen, R. J. (Ed.). (2013). *Theoretical Biology and Complexity: three essays on the natural philosophy of complex systems*. Academic Press.
- [21] Singh, A., Khan, R. A., Kushwaha, S., & Alshenqeeti, T. (2022). Roll of Newtonian and Non-Newtonian Motion in Analysis of Two-Phase Hepatic Blood Flow in Artery during Jaundice. *International Journal of Mathematics and Mathematical Sciences*, 2022.
- [22] Woodgerd, W. (1964). Population dynamics of bighorn sheep on Wildhorse Island. *The Journal of Wildlife Management*, 381-391.
- [23] <https://hankstevens.github.io/Primer-of-Ecology/DDgrowth.html#continuous-logistic-growth>