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On Vague Cosets and Vague Normal Subgroups

Tarun Kumar[1], Dhanpal Singh[2*], M. K. Sharma[3], Ajeet Singh[4], Pardeep Kumar[5]

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Abstract. In the present paper, some basic concepts of vague sets and vague groups together with the concepts of vague left and right cosets of a vague group were studied. To more appropriately introduce the vague normal subgroup, we made use of the feature of vague sets and, as a result, some characterizations of vague groups. We also gave some basic concepts related to vague cosets and on behalf of these concepts we proved some classical group theorems and we generalized some new theorems and their proofs on fuzzy coset and fuzzy normal subgroups to the vague coset and vague normal subgroups. We also discussed some important properties.

Keywords. *Vague Cosets,Vague Normal Subgroups ;Fuzzy Sets, Fuzzy Cosets, Fuzzy Subgroups.*

1. Introduction

In 1965, the concept of fuzzy sets was 1st developed by Zadeh [1]. Zadeh's work has unplugged new perceptions and applications with a wide range in the scientific fields. Many real-life problems are often not crip and this problem cannot be handled with the help of bi-valued logic i.e., yes or no. Later, Zadeh described a mathematical view on such kind of problems containing vagueness, by giving some truth value function known as membership function to each element of a given problem. Kaufmann [2] developed the notion of fuzzy subsets. In 1971, In his introduction to the idea of fuzzy groups, Rosenfield [3] also covered the notion of fuzzy subgroupoids and fuzzy groups. In 1979, Sherwood and Anthony [4] redescribed the fuzzy group and gave the theory of fuzzy group w.r.t. t-norm. In 1981, W.M. Wu [5] established the normal fuzzy subgroup and discussed some properties of the normal fuzzy subgroup. In 1984, N.P. Mukherjee [6] explained some properties of fuzzy costs along with fuzzy normal subgroups. As a fuzzy set theory extension, Buehrer and Gau [8] later established the idea of vague sets. The concept behind a vague set is that each element's membership can be divided into two categories: true membership and false membership. In 1994, N. Ajmal

[1]*Department of Mathematics, "Chaudhary Charan Singh University, Meerut (U.P.) India-250004, tkvats3@gmail.com*
[2*]*Department of Mathematics, Keshav Mahavidyalaya, University of Delhi, Delhi -110034,India, drdhanpalsingh@keshav.du.ac.in*
[3]*Department of Mathematics, "Chaudhary Charan Singh University, Meerut (U.P.) India-250004, mukeshkumarsharma@ccsuniversity.ac.in*
[4]*Department of Mathematics, "Chaudhary Charan Singh University, Meerut (U.P.) India-250004, singhajeet66@gmail.com*
[5]*Department of Mathematics, "Indraprastha College for Women,University of Delhi, Delhi,India, pardeep@ip.du.ac.in*
Corresponding Author:Dhanpal Singh
**Department of Mathematics, Keshav Mahavidyalaya, University of Delhi, Delhi -110034,India*

[7] proposed the lattice of the normal and fuzzy subgroups and discussed some properties of the lattice on the fuzzy normal subgroup. In 1999, Demirci [9] gave a mathematical concept related to Vague groups, which was a different theory from the theory of R. Biswas[12]. In 2001, Yunjie Zhang [10]2020 *Mathematics Subject Classification*. 03E72, 94D05;05B10,14L30;18C40gave some properties of fuzzy subgroups. In 2005, Wang Jue. [11] studied the roughness of vague sets. Biswas in [12, 13] proposed the idea of Vague groups (VGs) and examined some properties of VGs. Further, in 2008, Ramakrishna in [14, 15] studied normal groups, vague normalizers, vague centralizers, vague weights, and the characterization of cyclic groups with the help of vague groups. In 2011, A. Solairaju [16] studied Q-vague as well as vague normal subgroups w.r.t (T, S) norms. In 2015, V. Amarendra Babu [17] introduced the idea of vague additive groups with their properties. In 2016, Mallika and Ramakrishna [18] studied the concept of vague cosets, vague symmetry, vague invariant, and some of their important properties. Onasanya and ilori in [19, 20] discussed some properties of fuzzy cosets and normal groups. In 2017, Zelalem Teshome Wale [21], studied the concept of I-vague products. In 2020, N. Ramakrishna [22] introduced the notion of multi-vague algebra by defining multi-vague groups, (α,β)-cut of the multi-vague set, true -cut , false α-cut and studied some various properties in the multi-vague group. Nazimul [23] described some results on fuzzy subgroups and fuzzy cosets. Five sections make up the current study paper, including an introduction. In the second section, we outlined some of the fundamental concepts that were used in this work. In the third section, some basic preliminaries; vague sets, vague cosets, vague normal subgroups, and some propositions are given. In the fourth section, we have proved some theorems based on vague cosets and vague normal subgroups. In the next and fifth sections, we gave the conclusion of this work.

2. NOTATIONS

$V_A =$ vague set

$t_A =$ Truth membership function.

$f_A =$ False membership function.

$\forall =$ for all.

$\epsilon =$ belongs to.

$G =$ A group equipped with binary operation $*$.

$aV_A =$ vague left coset.

$V_A a =$ vague right coset.

3. Preliminary

Definition 3.1. *Vague set*

A vague set A in the universe of discourse X is characterized by two membership functions given by; (1) A truth membership function $t_A : X \to [0, 1]$ *(2) A false membership function* $f_A : X \to [0, 1]$ *here* $t_A(x)$ *indicates membership grade's lower bound of x obtained from "witness for* x *" and* $f_A(x)$ *represents "lower bound" on the x negation dervied from the "witness against x" and* $t_A(x) + f_A(x) \leq 1$. *Consequently, the membership grade of x in the "vague set A " is bounded with the sub-interval* $t_A(x)$, $1 - f_A(x)$ *of* $[0, 1]$. *The vague set A can be described as indicated in* $VA(x)$. = $<x, t_A(x),\ f_A(x)\ /x\ \in X$. *Where the interval* $[t_A(x), 1 - f_A(x)\ ]$ *is said to be the x value in the "vague set" A and*

Definition 3.2. *Vague Group*

Suppose G *to be a group,* AV_A *of* G *is considered to be a* VG *of* G, *if the following properties are true:* $-\forall x, y \in G$.

i. $V_A(xy) \geq \min\{V_A(x), V_A(y)\}$. *i.e.,* $t_A(xy) \geq \min\{t_A(x), t_A(y)\}$ $\&1 - f_A(xy) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$.

ii. $V_A\left(x^{-1}\right) \geq V_A(x)$. *i.e.,* $t_A\left(x^{-1}\right) \geq t_A(x) \& 1 - f_A\left(x^{-1}\right) \geq 1 - f_A(x)$. *here the element* (xy) *stand for* $(x * y)$. *and '*', is a binary operation defined on G.*

Definition 3.3. *Vague left coset*

Let V_A *be a vague group of a group G, for any* $a \in G$, *aG, a vague left coset of* V_A *is denoted by a* V_A *and defined by* $-$ $(aV_A)(x) = V_A a^{-1} x$ *i.e.,* $(at_A)(x) = t_A a^{-1} x \& 1 - af_A(x) = 1 - f_A a^{-1} x\ \forall x \in G$.

Definition 3.4. *Vague right coset*

Let V_A *be a vague group of a group G, any* $a \in G$, *a vague right coset of* V_A *is denoted by* $V_A(a)$ *and defined by;*

$$(V_A a)(x) = V_A x a^{-1} \text{ i.e. } (t_A a)(x) = t_A x a^{-1} \& 1 - f_A a(x) = 1 - f_A x a^{-1} \qquad \forall x \in G.$$

Definition 3.5. *Vague normal subgroup*

Let V_A *be a vague group of a group G then* V_A *is called Vague normal subgroup if* $\forall x, y \in G, V_A(xy) = V_A(yx)$ *i.e.* $t_A(xy) = t_A(yx) \& 1 - f_A(xy) = 1 - f_A(yx)$. *As an alternative, a vague group* V_A *is considered to be a vague normal subgroup of* G, *f*$V_A(x) = V_A y x y^{-1}$ *i.e.* $t_A(x) = t_A y x y^{-1} \& 1 - f_A(x) = 1 - f_A y x y^{-1} \forall x, y \in G$.

Proposition 3.6. *Let G be a group and A be any vague group of G; then* $V_A x^{-1} = V_A(x)\ \forall x \in G$.

Proposition 3.7. V_A *is a vague group of group G if and only if* $VAxy^{-1} \geq \min V_A(x), V_A(y) \qquad \forall x, y \in G$.

Proposition 3.8. *If* $V_A xy{-}^1 = V_A(e)$ *then* $V_A(x) = V_A(y)$ *is a vague group of G. where e indicates the identity of group* G *and* V_A *denotes a vague group of* G, $\forall x, y \in G$.

Proposition 3.9. *If* V_A *is a vague group of a group G then* $V_A(x) \leq V_A(e)\ \forall x \in G$.

4. THEOREMS AND THEIR MATHEMATICAL PROOFS

Theorem 4.1. *Let* V_A *be a vague group of a group G, then* $eV_A = V_A = V_A e$ *i.e. et* $t_A = t_A = t_A e$ *and* $1 - ef_A = 1 - f_A = 1 - f_A e$, *where e being identity in G.*

Proof: We assume that V_A to be a vague group of G and e being the identity within G. Then eV_A and $V_A e$ are respectively left and right cosets of V_A in G. Now, $\forall x \in G$, We have,

$$-(et_A)(x) = t_A e^{-1} x = -t_A(x) = t_A x e^{-1} = (t_A e)(x) \qquad \& 1 - ef_A(x) = 1 - f_A e^{-1} x$$

$= 1 - f_A(x) = 1 - f_A x e^{1} = 1 - f_A e(x) \forall x \in G$

$\Rightarrow etA = tA = tAe \& 1 - efA = 1 - fA = 1 - fAe$. $\Rightarrow eVA = VA = VAe$.

Theorem 4.2. *Let* V_A *be a vague group of a group G and* $a \in G$. *Then,* $V_A(a) = V_A(e)$ *i.e.* $t_A(a) = t_A(e)$ *and* $1 - f_A(a) = 1 - f_A(e)$ *if and only if* $aV_A = V_A = V_A a$ *i.e.* $at_A = t_A = t_A a$ *and* $1 - af_A = 1 - f_A = 1 - f_A a$ *where e being identity in G.*

Proof: Let $a \in G$ with $V_A(a) = V_A(e)$ i.e. $t_A(a) = t_A(e)$ *and* $1 - f_A(a) = 1 - f_A(e)$. Now, $\forall x \in G$. We have $(at_A)(x) = t_A\left(a^{-1}x\right) \geq \min(t_A(a), t_A(x)) = \min(t_A(e), t_A(x)) = t_A(x)$ *and* $1 - (af_A)(x) = 1 - f_A\left(a^{-1}x\right) \geq \min(1 - f_A(a), 1 - f_A(x)) = \min(1 - f_A(e), 1 - f_A(x)) = 1 - f_A(x)$. [Since, V_A is a vague group and $V_A(x) \leq V_A(e)$] again, $\forall x \in G$. We have $t_A(x) = t_A\left(aa^{-1}x\right) \geq \min\left(t_A(a), t_A\left(a^{-1}x\right)\right) = \min\left(t_A(e), t_A\left(a^{-1}x\right)\right) = t_A\left(a^{-1}x\right) = (at_A)(x)$ *and* $1 - f_A(x) = 1 - f_A\left(aa^{-1}x\right) \geq \min\left(1 - f_A(a), 1 - f_A\left(a^{-1}x\right)\right) = \min(1 - f_A(e), 1 - f_A\left(a^{-1}x\right)) = 1 - af_A(x)$. [Since, V_A is a vague group and $V_A(x) \leq V_A(e)$]. We conclude that from above $at_A = t_A \& 1 - af_A = 1 - f_A \Rightarrow aV_A = V_A$. Similiarly, we can prove that $t_A = t_A a \& 1 - f_A = 1 - f_A a \Rightarrow V_A = V_A a$. Conversly, Let $at_A = t_A = t_A a$. Consider, $t_A(a) = (t_A a)(a) = t_A\left(aa^{-1}\right) = t_A(e)$ *and* , $1 - f_A(a) = 1 - (f_A a)(a) = 1 - f_A\left(aa^{-1}\right) = 1 - f_A(e) \Rightarrow V_A(a) = V_A(e)$.

Theorem 4.3. *Let V_A be a normal vague group of a group G and $a \in G$,then every left coset of V_A is a right coset of V_A in G i.e. $aV_A = V_A a$ i.e. $at_A = t_A a$ and $1 - af_A = 1 - f_A a$ $\forall x \in G$.*

Proof: Let V_A be a normal vague subgroup of a group G and $a \in G$. Then aV_A and $V_A a$ are respectively left and right coset of V_A in G. Now, $\forall x \in G$, we have $(at_A)(x) = t_A\left(a^{-1}x\right) = t_A\left(xa^{-1}\right) = (t_A a)(x)$ and $1-(af_A)(x) = 1-f_A\left(a^{-1}x\right) = 1 - f_A\left(xa^{-1}\right) = 1 - (f_A a)(x) \Rightarrow (at_A)(x) = (t_A a)(x) \& 1 - (af_A)(x) = 1 - (f_A a)(x) \Rightarrow aV_A = V_A a \; \forall a \in G$

Theorem 4.4. *Let V_A be a vague group of an abelian group G and $a \in G$, then every left coset of V_A is a right coset of V_A in G i.e. $aV_A = V_A a$ i.e. $at_A = t_A a$ and $1 - af_A = 1 - f_A a$ $\forall x \in G$.*
Proof: Proof of theorem 4.3 could be followed.

Theorem 4.5. *Let V_A and V_B be two vague groups of a group* **G** *and $a \in G$. Then- (i) $aV_A = aV_B$ if and only if $V_A = V_B$ i.e., $at_A = at_B$ if and only if $t_A = t_B$ and $1 - af_A = 1 - af_B$ if and only if $1 - f_A = 1 - f_B$. (ii) $V_A a = V_B a$ if and only if $V_A = V_B$ i.e., $t_A a = t_B a$ if and only if $t_A = t_B$ and $1 - f_A a = 1 - f_B a$ if and only if $1 - f_A = 1 - f_B$.*

Proof: Let $aV_A = aV_B$, where $a \in G$ and let e being identity in G, $\Rightarrow$a $t_A = at_B$ *and* $1 - af_A = 1 - af_B$. From the definition of left coset, we have $\forall x \in G, t_A(x) = t_A(ex) = t_A\left(a^{-1}ax\right) = (at_A)(ax) = (at_B)(ax) = t_B\left(a^{-1}ax\right) = t_B(ex) = t_B(x)$ *and* $1 - f_A(x) = 1 - f_A(ex) = 1 - f_A\left(a^{-1}ax\right) = 1 - (af_A)(ax) = 1 - (af_B)(ax) = 1 - f_B\left(a^{-1}ax\right) = 1 - f_B(ex) = 1 - f_B(x) \Rightarrow t_A(x) = t_B(x)$ *and* $1 - f_A(x) = 1 - f_B(x) \Rightarrow V_A(x) = V_B(x) \Rightarrow V_A = V_B$. Conversly, suppose that $V_A = V_B$ i.e. $t_A = t_B$ *and* $1 - f_A = 1 - f_B$. Now, $\forall x \in G$, $(at_A)(x) = t_A\left(a^{-1}x\right) = t_B\left(a^{-1}x\right) = (at_B)(x)$ *and* $1 - (af_A)(x) = 1 - f_A\left(a^{-1}x\right) = 1 - f_B\left(a^{-1}x\right) = 1 - (af_B)(x) \Rightarrow at_A = at_B \& 1 - af_A = 1 - af_B \Rightarrow aV_A = aV_B$.
(ii) Proof: - of part (i) can be followed.

Theorem 4.6. *Let V_A be a vague group of a group G nd $a^{-} \in G$. If $V_A = V_A a^{-1}xa$ i.e. $t_A(x) = t_A a^{-1}xa$ and $1 - f_A(x) = 1 - f_A a^{1}xa \forall x \in G$ then $aV_A = V_A a$ i.e., $at_A = t_A a$ and $1 - af_A = 1 - f_A a$.*

Proof: Let V_A be a vague group of a group G and let $a \in G$ and e being identity in G. Let $V_A = V_A\left(a^{-1}xa\right)$ $\forall x \in G$ i.e. $t_A = t_A\left(a^{-1}xa\right)$ *and* $1 - f_A = 1 - f_A\left(a^{-1}xa\right)$. Now, from the definition of right coset, we have $\forall x \in G, (t_A a)(x) = t_A\left(xa^{-1}\right) = t_A\left(a^{-1}\left(xa^{-1}\right)a\right) = t_A\left(a^{-1}xa^{-1}a\right) = t_A\left(a^{-1}xe\right) = t_A\left(a^{-1}x\right) = (at_A)(x)$ *and* $1-(f_A a)(x) = 1-f_A\left(xa^{-1}\right) = 1-f_A\left(a^{-1}\left(xa^{-1}\right)a\right) = 1-f_A\left(a^{-1}xa^{-1}a\right) = 1-f_A\left(a^{-1}xe\right) = 1-f_A\left(a^{-1}x\right) = 1 - (af_A)(x)$. $\Rightarrow t_A a = at_A$ *and* $1 - f_A a = 1 - af_A \Rightarrow aV_A = V_A a$

Theorem 4.7. *Let V_A be a vague group of a group* G *and* ***a***,***b***,***c***,***d*** *$\in G$. Then, i. $a\,(bV_A) = (ab)V_A$, if bV_A is a vague group of G. ii.$(V_A a)\,b = V_A(ab)$, if $V_A a$ is a vague group of G. iii. $(ab)V_A(cd) = a\,(bV_A c)\,d$, if $bV_A c$ is a vague group of G.*

Proof: (i) $\forall x \in G$, We have, $[a(bt_A)](x) = (bt_A)\left(a^{-1}x\right) = t_A\left(b^{-1}\left(a^{-1}x\right)\right) = t_A\left(b^{-1}a^{-1}x\right) = t_A\left((ab)^{-1}x\right) = [(ab)t_A](x)$ *and* $1-[a(bf_A)](x) = 1-(bf_A)\left(a^{-1}x\right) = 1-f_A\left(b^{-1}\left(a^{-1}x\right)\right) = 1-f_A\left(b^{-1}a^{-1}x\right) = 1-f_A\left((ab)^{-1}x\right) = 1-[(ab)f_A](x) \Rightarrow [a(bt_A)](x)$ $=[(ab)t_A](x) \& 1-[a(bf_A)](x) = 1-[(ab)f_A](x). \Rightarrow a(bV_A) = (ab)V_A.$

(ii) $\forall x \in G$, we have, $[(t_Aa)b](x) = (t_Aa)\left(xb^{-1}\right) = t_A\left(xb^{-1}a^{-1}\right) = t_A\left(x(ab)^{-1}\right) = [t_A(ab)](x)$ *and* $1-[(f_Aa)b](x) = 1-(f_Aa)\left(xb^{-1}\right) = 1-f_A\left(xb^{-1}a^{-1}\right) = 1-f_A\left(x(ab)^{-1}\right) = 1-[f_A(ab)](x) \Rightarrow [(t_Aa)b](x) = [t_A(ab)](x)$ *and* $1-[(f_Aa)b](x) = 1-[f_A(ab)](x)$.

(iii) $\forall x \in G$, we have, $[(ab)t_A(cd)](x) = t_A\left[(ab)^{-1}x(cd)^{-1}\right] = t_A\left[b^{-1}a^{-1}xd^{-1}c^{-1}\right] = t_A\left[b^{-1}\left(a^{-1}xd^{-1}\right)c^{-1}\right. = (bt_Ac)\left(a^{-1}xd^{-1}\right) = [a(bt_Ac)d](x)$ *and* $1-[(ab)f_A(cd)](x) = 1-f_A\left[(ab)^{-1}x(cd)^{-1}\right] = 1-f_A\left[b^{-1}a^{-1}xd^{-1}c^{-1}\right] = 1-f_A\left[b^{-1}\left(a^{-1}xd^{-1}\right)c^{-1}\right. = 1-(bf_Ac)\left(a^{-1}xd^{-1}\right) = 1-[a(bf_Ac)d](x). \Rightarrow [(ab)t_A(cd)](x) = [a(bt_Ac)d](x)$ *and* $1-[(ab)f_A(cd)](x) = 1-[a(bf_Ac)d](x). \Rightarrow)\ (ab)V_A(cd) = a(bV_Ac)d.$

Theorem 4.8. *Let V_A and V_B be two vague groups of a group G and $a,b \in G$. Then,(i) $aV_A = bV_B$ if and only if $V_A = a^{-1}bV_B$ i.e. $at_A = bt_B$ if and only if $t_A = a^{-1}bt_B$ and $1-af_A = 1-bf_B$ if and only if $1-f_A = 1-a^{-1}bf_B$. (ii) $V_Aa = V_Bb$ if and only if $V_A = V_Bba^{-1}$ i.e. $t_Aa = t_Bb$ if and only if $t_A = t_Bba^{-1}$ and $1-f_Aa = 1-f_Bb$ if and only if $1-f_A = 1-f_Bba^{-1}$.*

Proof: (i) Let V_A and V_B be two vague groups of a group G and $a,b \in G$ and e being identity in G. Suppose that $-aV_A = bV_B$ i.e. $at_A = bt_B$ *and* $1-af_A = 1-bf_B$. Now $\forall x \in G$, We have $\left(a^{-1}bt_B\right)(x) = t_B\left(\left(a^{-1}b\right)^{-1}x\right) = t_B\left(b^{-1}\left(a^{-1}\right)^{-1}x\right) = t_B\left(b^{-1}ax\right) = bt_B(ax) = at_A(ax) = t_A\left(a^{-1}ax\right) = t_A(ex) = t_A(x)$ *and* $1-\left(a^{-1}bf_B\right)(x) = 1-f_B\left(\left(a^{-1}b\right)^{-1}x\right) = 1-f_B\left(b^{-1}\left(a^{-1}\right)^{-1}x\right) = 1-f_B\left(b^{-1}ax\right) = 1-bf_B(ax) = 1-af_A(ax) = 1-f_A\left(a^{-1}ax\right) = 1-f_A(ex) = 1-f_A(x). \Rightarrow t_A = a^{-1}bt_B$ *and* $1-f_A = 1-a^{-1}bf_B \Rightarrow V_A = a^{-1}bV_B$.

Conversly, suppose that $V_A = a^{-1}bV_B$ i.e. $t_A = a^{-1}bt_B$ *and* $1-f_A = 1-a^{-1}bf_B$. Now, $\forall x \in G$, we have $(at_A)(x) = t_A\left(a^{-1}x\right) = \left(a^{-1}bt_B\right)\left(a^{-1}x\right) = t_B\left(\left(a^{-1}b\right)^{-1}\left(a^{-1}x\right)\right) = t_B\left(b^{-1}\left(a^{-1}\right)^{-1}\left(a^{-1}x\right)\right) = t_B\left(b^{-1}aa^{-1}x\right) = t_B\left(b^{-1}ex\right) = t_B\left(b^{-1}x\right) = bt_B(x)$, *and* $1-(af_A)(x) = 1-f_A\left(a^{-1}x\right) = 1-\left(a^{-1}bf_B\right)\left(a^{-1}x\right) = 1-f_B\left(\left(a^{-1}b\right)^{-1}\left(a^{-1}x\right)\right) = 1-f_B\left(b^{-1}\left(a^{-1}\right)^{-1}\left(a^{-1}x\right)\right) = 1-f_B\left(b^{-1}aa^{-1}x\right) = 1-f_B\left(b^{-1}ex\right) = 1-f_B\left(b^{-1}x\right) = 1-bf_B(x). \Rightarrow at_A = bt_B$ *and* $1-af_A = 1-bf_B. \Rightarrow aV_A = bV_B$.

(ii) same as above.

Theorem 4.9. *Let V_A be a vague group of a group G nd $a,b,c,d \in G$. Then, $aV_Ab = cV_Ad$ if and only if $b^{-1}V_Aa^{-1} = d^{-1}V_Ac^{-1}$ i.e., $at_Ab = ct_Ad$ if and only if $b^{-1}t_Aa^{-1} = d^{-1}t_Ac^{-1}$ and $1-af_Ab = 1-cf_Ad$ if and only if $1-b-1\ fAa-1 = 1-d-1\ fAc-1$*

Proof: Let $aV_Ab = cV_Ad$ i.e. $at_Ab = ct_Ad$ *and* $1-af_Ab = 1-cf_Ad$. Now, $\forall x \in G$, we have, $\left(b^{-1}t_Aa^{-1}\right)(x) = t_A(bxa) = t_A\left[(bxa)^{-1}\right] = t_A\left[a^{-1}x^{-1}b^{-1}\right] = (at_Ab)\left(x^{-1}\right) = (ct_Ad)\left(x^{-1}\right) = t_A\left(c^{-1}x^{-1}d^{-1}\right) = t_A\left[(dxc)^{-1}\right] = t_A(dxc) = \left(d^{-1}t_Ac^{-1}\right)(x)$ *and* $1-\left(b^{-1}f_Aa^{-1}\right)(x) = 1-f_A(bxa) = 1-f_A\left[(bxa)^{-1}\right] = 1-f_A\left[a^{-1}x^{-1}b^{-1}\right] = 1-(af_Ab)\left(x^{-1}\right) = 1-(cf_Ad)\left(x^{-1}\right) = 1-f_A\left(c^{-1}x^{-1}d^{-1}\right) = 1-f_A\left[(dxc)^{-1}\right] = 1-f_A(dxc) = 1-\left(d^{-1}f_Ac^{-1}\right)(x)$.

$\Rightarrow \left(b^{-1}t_Aa^{-1}\right)(x) = \left(d^{-1}t_Ac^{-1}\right)(x)$ *and* $1-\left(b^{-1}f_Aa^{-1}\right)(x) = 1-\left(d^{-1}f_Ac^{-1}\right)(x) \Rightarrow b^{-1}V_Aa^{-1} = d^{-1}V_Ac^{-1}$

Conversly, suppose that $b^{-1}V_Aa^{-1} = d^{-1}V_Ac^{-1}$ i.e. $\left(b^{-1}t_Aa^{-1}\right)(x) = \left(d^{-1}t_Ac^{-1}\right)(x)$ *and* $1-\left(b^{-1}f_Aa^{-1}\right)(x) = 1-\left(d^{-1}f_Ac^{-1}\right)(x)$. Now $\forall x \in G$, we have, $(at_Ab)(x) = t_A\left(a^{-1}xb^{-1}\right) = t_A\left[\left(a^{-1}xb^{-1}\right)^{-1}\right] = t_A\left[\left(b^{-1}\right)^{-1}x^{-1}\left(a^{-1}\right)^{-1}\right] = t_A\left[bx^{-1}a\right] = \left(b^{-1}t_Aa^{-1}\right)\left(x^{-1}\right) = \left(d^{-1}t_Ac^{-1}\right)\left(x^{-1}\right) = t_A\left[dx^{-1}c\right] = t_A\left[\left(dx^{-1}c\right)^{-1}\right] = t_A\left[c^{-1}xd^{-1}\right] = (ct_Ad)(x)$ · *and* $1-(af_Ab)(x) = 1-f_A\left(a^{-1}xb^{-1}\right) = 1-f_A\left[\left(a^{-1}xb^{-1}\right)^{-1}\right] = 1-f_A\left[\left(b^{-1}\right)^{-1}x^{-1}\left(a^{-1}\right)^{-1}\right] = 1-f_A\left[bx^{-1}a\right] = 1-\left(b^{-1}f_Aa^{-1}\right)\left(x^{-1}\right) = 1-\left(d^{-1}f_Ac^{-1}\right)\left(x^{-1}\right) = 1-f_A\left[dx^{-1}c\right] = 1-f_A\left[\left(dx^{-1}c\right)^{-1}\right] = 1-f_A\left[c^{-1}xd^{-1}\right] = 1-(cf_Ad)(x). \Rightarrow (at_Ab)(x) = (ct_Ad)(x) \Rightarrow 1-(af_Ab)(x) = 1-(cf_Ad)(x) \Rightarrow aV_Ab = cV_Ad$

Theorem 4.10. *Let V_A be a vague group of a group G. Let H be a normal subgroup of G and $V_A(x) = V_A(e)$; $\forall x \in H$ then $\exists a$ function $A_H : \frac{G}{H} \longrightarrow [0, 1]$ such that $V_{AH} = \frac{G}{H}, A_H$ is a vague group.*

Proof: Since, H is normal subgroup of G.Therefore, $\frac{G}{H}$ is a group. Now, define a mapping

$$A_H : \frac{G}{H} \to [0,1] such that$$

$$V_{A_H}(xH) = \begin{cases} V_A(e); x \in H \\ V_A(x); x \notin H \end{cases}$$

$$i.e., t_{A_H}(xH) = \begin{cases} t_A(e); x \in H \\ t_A(x); x \notin H \end{cases} \& 1 - f_{A_H}(xH) = \begin{cases} 1 - f_A(e); x \in H \\ 1 - f_A(x); x \notin H \end{cases}$$

Now, assume that, $xH = yH \Rightarrow xy^{-1} \in H$. *Consider,* $V_A\left(xy^{-1}\right) = V_A(e) \Rightarrow V_A(x) = V_A(y) \Rightarrow V_{A_H}(xH) = V_{A_H}(yH)$ *i.e.t* $t_{A_H}(xH) = t_{A_H}(yH)$ *and* $1 - f_{A_H}(xH) = 1 - f_{A_H}(yH)$. *Hence,* V_{A_H} *is a well-defined map. Now, if* $x, y \notin H$ *then consider,* $t_{A_H}\left((xH)^{-1}\right) = t_{A_H}\left(x^{-1}H\right) = t_A\left(x^{-1}\right) = t_A(x) = t_{A_H}(xH) \& 1 - f_{A_H}\left((xH)^{-1}\right) = 1 - f_{A_H}\left(x^{-1}H\right) =$

$1 - f_A\left(x^{-1}\right) = 1 - f_A(x) = 1 - f_{A_H}(xH). \Rightarrow V_{A_H}\left((xH)^{-1}\right) = V_{A_H}(xH)$. *Also,* $t_{A_H}(xH * yH) = t_{A_H}(xyH) = t_A(xy) \geq \min(t_A(x), t_A(y)) \geq \min(t_{A_H}(x), t_{A_H}(y)) \geq \min(t_{A_H}(xH), t_{A_H}(yH))$ *and* $1 - f_{A_H}(xH * yH) = 1 - f_{A_H}(xyH) = 1 - f_A(xy) \geq \min(1 - f_A(x), 1 - f_A(y)) \geq \min(1 - f_{A_H}(x), 1 - f_{A_H}(y)) \geq \min(1 - f_{A_H}(xH), 1 - f_{A_H}(yH)) \Rightarrow V_{A_H}(xH * yH) \geq \min(V_{A_H}(xH), V_{A_H}(yH))$. *If* $x, y \in H$ *then consider* , $t_{A_H}\left((xH)^{-1}\right) = t_{A_H}\left(x^{-1}H\right) = t_A(e) = t_{A_H}(xH)$ *and* $1 - f_{A_H}\left((xH)^{-1}\right) = 1 - f_{A_H}\left(x^{-1}H\right) = 1 - f_A(e) = 1 - f_{A_H}(xH). \Rightarrow V_{A_H}\left((xH)^{-1}\right) = V_{A_H}(xH)$. *Also,* $t_{A_H}(xH * yH) = t_{A_H}(xyH) = t_A(e) \geq \min(t_A(x), t_A(y)) \geq \min(t_{A_H}(x), t_{A_H}(y)) \geq \min(t_{A_H}(xH), t_{A_H}(yH)) \& 1 - f_{A_H}(xH * yH) = 1 - f_{A_H}(xyH) = 1 - f_A(e) \geq \min(1 - f_A(x), 1 - f_A(y)) \geq \min(1 - f_{A_H}(x), 1 - f_{A_H}(y)) \geq \min(1 - f_{A_H}(xH), 1 - f_{A_H}(yH)) \Rightarrow V_{A_H}(xH * yH) \geq \min(V_{A_H}(xH), V_{A_H}(yH))$.

5. Conclusion

Group theory has many real-life applications in the field of Space study, Analytical chemistry, computer science, medical sciences, life sciences, economics, etc. Here, we studied some new outcomes of vague groups and then defined a novel way to represent the vague left and right cosets. The idea of a vague normal subgroup has also been defined. The basic objective of this paper is to find outsome new outcomes on vague coset and vague normal groups. We generalized the results of fuzzy group theory to the vague group theory by adding the false membership value in existing results of fuzzy group theory and also in the last theorem, we provedthe existence of a function for making crisp quotient group a vague group and found that these results are applicable in vague group theory.Vague set theory was universallyused in pattern recognition,decision-making, logic programming,andmedical diagnosis. It seems to have been more popular than fuzzy sets technology in recent years. In the end, we hope that this work encourages further study in this field. As it is the area has maximum application in almost every field.

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