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Connected Certified Domination Number of Splitting Graphs of Certain Graphs

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Abstract: A dominating set S of a graph $G = (V, E)$ is called a certified dominating set of G. If every vertices in S has either zero or at least two neighbours in $V(G) - S$. A certified dominating set S of G is said to be connected certified dominating set if the subgraph induced by S is connected. The minimum cardinality taken over all the connected certified dominating set is called the connected certified domination number of G and is denoted by $\gamma_{cer}^{c}(G)$. In this paper, we investigate the connected certified domination number of splitting graphs of certain graphs.

Keywords: Dominating set, certified dominating set, certified domination number, connected certified domination, Splitting graphs.

AMS: 05C69

1. Introduction

Let $G = (V, E)$ be a finite, undirected graph without loops and multiple edges. The graph G has $n = |V|$ vertices and $m = |E|$ edges. A path P_n is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i v_{i+1}\}$, where $i = 1,2, \ldots, n-1$. A cycle is a path from a vertex back to itself (So the first and last vertices are not distinct). A complete graph K_n is a graph in which any two distinct vertices are adjacent. A complete bipartite graph, denoted by $K_{m,n}$ is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y. A star is a complete bipartite graph $K_{1,n}$. The join $G + H$ of graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv; u \in V(G)\ and\ v \in V(H)\}$. The fan graph of order n is defined as $K_1 + P_n$ and is denoted by F_n or $F_{1,n}$. The wheel graph of order $n \geq 3$ is defined as $K_1 + C_n$ and is denoted by W_n or $W_{1,n}$.

Domination in graphs is one of the interesting areas in graph theory which has wide applications in Engineering and Science. There are more than 300 domination parameters available in the literature. Around 1960 Berge and ore started the mathematical exploration of domination theory in graphs. There is a plethora of material on domination theory; we recommend readers outstanding books [2,3] on domination-related parameters.

Suppose that we are given a group of X officials and a group of Y civilians. There $x \in X$ for each civil $y \in Y$ who can attend x, and every time any such y is attending x, there must be also another civil $z \in Y$ that observes y. That is z must act as a kind of witness, to sidestep any mismanagement from y. In the case of a certain social network, what is the minimum number of connected officials necessary to ensure such a service? This aforementioned issue motivates us to propose the concept of connected certified domination.

The theory of certified domination was introduced by Dettlaff, Lemanska, Topp, Ziemann and Zylnski [9] and further studded in [8]. It has many applications in real life situations. The concept

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of connected certified domination was introduced by A. Ilyass and V.S.Goswami[10]. This motivated we to study the connected certified number in central graphs of certain standard graphs such as complete, complete bipartite graph, path graph, cycle graph, wheel graph, fan graph and double star graph.

In [9], authors studied certified domination number in graphs which is defined as follows:

Definition 1.1

Let $G = (V, E)$ be any graph of order n. A subset $S \subseteq V(G)$ is said to be a certified dominating set of G if S is a dominating set of G and every vertex in S has either zero or at least two neighbours in $V - S$. The certified domination number denoted by $\gamma_{cer}(G)$ is the minimum cardinality of certified dominating sets in G.

Definition 1.2.

Let $G = (V, E)$ be any connected graph of order n. A certified dominating set $S \subseteq V(G)$ is called a connected certified dominating set of G if its induced subgraph $G[S]$ is connected. The connected certified domination number is the minimum cardinality of a connected dominating set of G and we denoted it by $\gamma_{cer}^{c}(G)$.

2. Preliminaries

Theorem 2.1 [9] For any graph G of order $n \geq 2$, every certified dominating set of G contains its support vertices.

Theorem 2.2 [9] For any graph G of order n, $1 \leq \gamma_{cer}^{c}(G) \leq n$.

Observation 2.3 [10]

1) Let $K_{m,n}$ be a complete bipartite graph, then $\gamma_{cer}^{c}(K_{m,n}) = 2$ for $3 \leq m \leq n$.
2) Let $K_{1,n-1}$ be a star graph, then $\gamma_{cer}^{c}(K_{1,n}) = 1$ for $n \geq 2$.
3) Let W_n be a wheel graph, then $\gamma_{cer}^{c}(W_n) = 1$.
4) Let $S_{1,n,n}$ be a double star graph, then $\gamma_{cer}^{c}(S_{1,n,n}) = 2$, where $n \geq 2$.

Observation 2.4 [10]

1) If K_n is a complete graph, then $\gamma_{cer}^{c}(K_n) = 1$ for $n \geq 3$.
2) If P_n is a path graph, then $\gamma_{cer}^{c}(P_n) = n$ for $n \geq 4$.
3) If C_n is a cycle graph, then $\gamma_{cer}^{c}(C_n) = n$ for $n \geq 4$.
4) If F_n is a fan graph, then $\gamma_{cer}^{c}(F_n) = 1$ for $n \geq 3$.

Observation 2.5 [10]

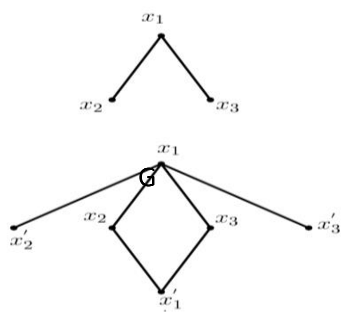
For any connected graph G, $\gamma_{cer}(G) \leq \gamma_{cer}^{c}(G)$.

3. Splitting Graphs

Definition: 3.1 [5]

The splitting graph $S'(G)$ of a connected graph G is obtained by adding a new vertex x' corresponding to each vertex x of G such that x' is adjacent to every vertex adjacent to x of G. If n is the order of G, then the order of $S'(G)$ is 2n. We call the vertices $x_1, x_2, \ldots, x_n$ are duplicated by $x_1', x_2', \ldots, x_n'$.

Example 3.2: Consider the graph G and their corresponding splitting graph $S'(G)$ is given in Figure 3.1



S’(G)Figure 3.1

Here $S = \{x_1', x_1\}$ is the unique minimum certified dominating set of $S'(G)$ and so $\gamma_{cer}(S'(G)) = 2$. But $< S >$ is not connected so that S is not a connected certified dominating set of $S'(G)$.

Since $x_1 - x_1'$ is connected by a path through x_2 and x_3, either x_2 or x_3 must be in a connected certified dominating set say S_1. If $S_1' = S \cup \{x_2\}$, then $x_1' \in S_1$ has exactly one neighbor in $V(S'(G)) - S_1$.

Consider $S_1 = S \cup \{x_2, x_3\}$. Clearly S_1 is a minimum connected certified dominating set of $S'(G)$ and so $\gamma_{cer}^c(S'(G)) = 4$.

Furthermore, in figure 3.1, $\gamma_{cer}^c(G) = 1$. Thus the connected certified dominating set of G and $S'(G)$ are different.

Observation 3.3

For any connected graph G of order n, $\gamma_c(G) \leq \gamma_c(S'(G))$.

Observation 3.4

For any connected graph G, there is no obvious relation connecting $\gamma_{cer}^c(G)$ and $\gamma_{cer}^c(S'(G))$.

Example for $\gamma_{cer}^c(S'(G)) < \gamma_{cer}^c(G)$.

Consider the connected graph G given in Figure 3.2

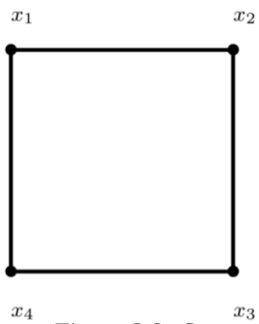


Figure 3.2: G

Here $S = \{x_1, x_2, x_3, x_4\}$ is the unique minimum connected certified dominating set of G and here, $\gamma_{cer}^c(G) = 4$.

Now we split every edge of G. A new connected graph $S'(G)$ obtained and given in Figure 3.3

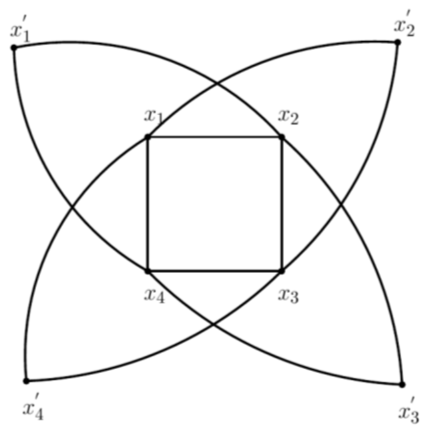


Figure 3.3: $S'(G)$

Here $S_1 = \{x_1, x_2, x_3\}$ is a minimum connected certified dominating set of $S'(G)$ and hence, $\gamma_{cer}^c(S'(G)) = 3$.

In this case, $\gamma_{cer}^c(S'(G)) < \gamma_{cer}^c(G)$. Example for $\gamma_{cer}^c(S'(G)) = \gamma_{cer}^c(G)$.

Consider the graph G given in Figure 3.4

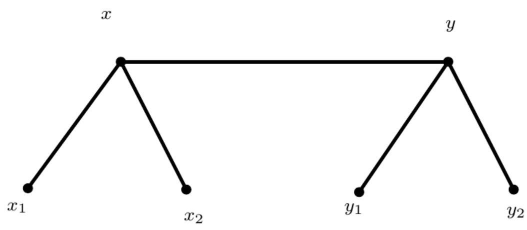


Figure 3.4: G

Here, $S = \{x, y\}$ is the unique minimum connected certified dominating set of G and hence, $\gamma_{cer}^{c}(G) = 2$.

Now we split every edge of G. A new connected graph $S'(G)$ obtained and is given in Figure 3.5

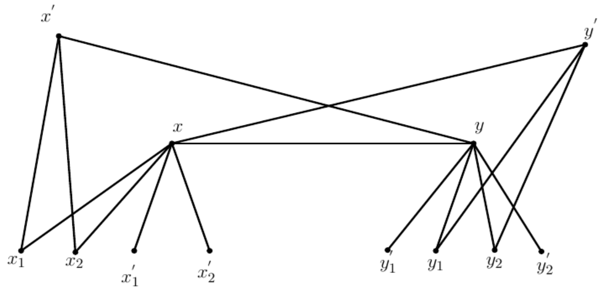


Figure 3.5: $S'(G)$

Here $S_1 = \{x, y\}$ is the unique minimum certified dominating set of $S'(G)$ and hence $\gamma_{cer}^{c}\big(S'(G)\big) = 2$.

Thus, in this case $\gamma_{cer}^{c}\big(S'(G)\big) = \gamma_{cer}^{c}(G)$.

Example for $\gamma_{cer}^{c}\big(S'(G)\big) > \gamma_{cer}^{c}(G)$.

Consider the graph G given in Figure 3.6

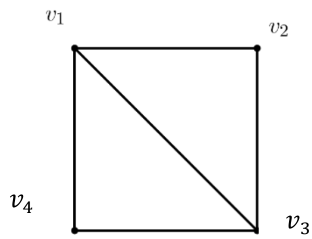


Figure 3.6: G

Here $S = \{v_1\}$ or $S = \{v_3\}$ is the minimum connected certified dominating set of G and hence, $\gamma_{cer}^{c}(G) = 1$.

Now, we split every edge of G. A new connected graph $S'(G)$ obtained an is given in Figure 3.7

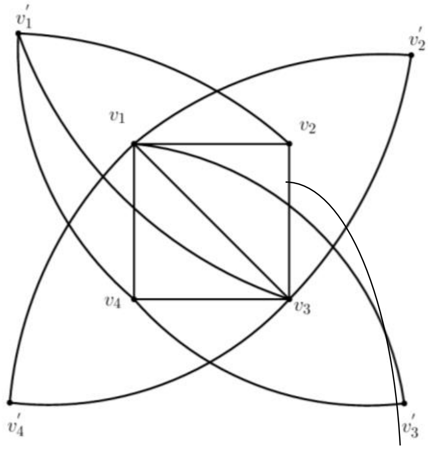


Figure 3.7: $S'(G)$

Here $S_1 = \{v_1, v_2\}$ is a minimum connected certified dominating set of $S'(G)$ and hence $\gamma_{cer}(S'(G)) = 2$.

Thus, in this case, $\gamma_{cer}^{c}(S'(G)) > \gamma_{cer}^{c}(G)$.

Theorem 3.5

Every support vertex of G being to every connected certified dominating set of $S'(G)$.

Proof:

Let G be a connected graph and let x be a support vertex of G. then there exists atleast one end vertex of G which is adjacent to x in G. we let it them be y. By the definition of $S'(G)$, y have a duplicated vertex y', which is adjacent to x in $S'(G)$. Clearly y' is an end vertex in $S'(G)$.

Let S be a connected certified dominating set in $S'(G)$. To prove $x \in S$. Suppose $x \notin S$. Then y' is not dominated by any vertex of S in $S'(G)$. This shows that $y' \in S \; in \; S'(G)$. Since $\deg(y') = 1$, that y' has exactly one neighbor x in $S'(G)$ and also y' has no adjacent vertex in S. This implies that S is not a connected certified dominating set of $S'(G)$, which is a contradiction. Hence $x \in S$in $S'(G)$.

Theorem 3.6

For any integer $n \geq 3, \gamma_{cer}^{c}\left(S'\left(K_{1,n-1}\right)\right) = 3$.

Proof:

Let $x_1, x_2, \ldots, x_{n-1}$ be the end vertices and x be the central vertex of the star $K_{1,n-1}$. Let $x', x_1', x_2', \ldots, x_{n-1}'$ be the corresponding duplicated vertices of $x, x_1, x_2, \ldots, x_{n-1}$, respectively to form $S'\left(K_{1,n-1}\right)$ and is given in Figure 3.8

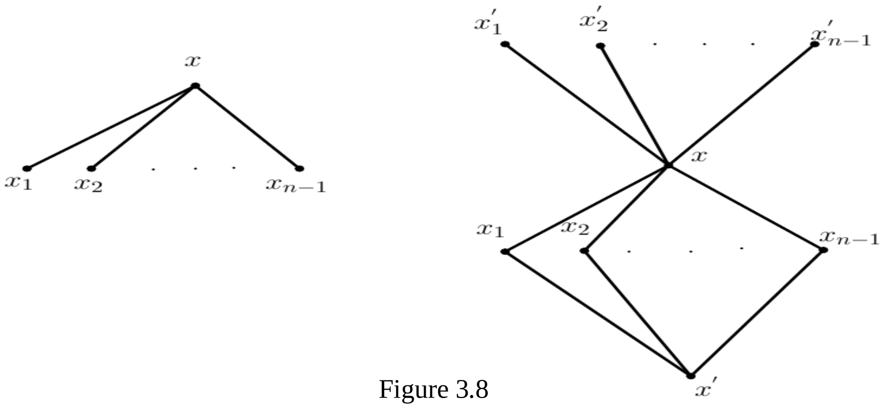


Figure 3.8

Clearly, $V\left(S'\left(K_{1,n-1}\right)\right)=\{x,x_1',x',x_i/1\le i\le n-1\}$ and so $\left|V\left(S'\left(K_{1,n-1}\right)\right)\right|=2n$. Since x is a support vertex of $K_{1,n-1}$ by Theorem 3.5, x must belong to every connected certified dominating set of $S'\left(K_{1,n-1}\right)$. Also $\{x\}$ dominates $V(S'\left(K_{1,n-1}\right))\{x'\}$. Therefore $S=\{x,x'\}$ forms a certified dominating set of $S'\left(K_{1,n-1}\right)$ because x and x' has more than two neighbor in $V\left(S'\left(K_{1,n-1}\right)\right)\backslash S$. But $< S >$ is not connected. So that S is not a connected certified dominating set of $S'\left(K_{1,n-1}\right)$. Now, consider $S_1=S\cup\{x_i\}$ for $1\le i\le n-1$. Since x_i is adjacent with x and x' in $S'\left(K_{1,n-1}\right)$, that $< S_1 >$ is connected. Therefore, S_1 is a connected certified dominating set of $S'\left(K_{1,n-1}\right)$ and so $\gamma_{cer}^{c}\left(S'\left(K_{1,n-1}\right)\right)\ge|S_1|=3$. Moreover, there does not exist a connected certified dominating set of cardinality less than 2, we conclude that $\gamma_{cer}^{c}\left(S'\left(K_{1,n-1}\right)\right)=3$.

Example 3.7

For the connected graph $S'\left(K_{1,6}\right)$given in Figure 3.9, by Theorem 3.6, $S=\{x,x',x_i\}$, for $1\le i\le 6$ is a minimum connected certified dominating set of $S'\left(K_{1,6}\right)$ and hence, $\gamma_{cer}^{c}\left(S'\left(K_{1,6}\right)\right)=|S|=3$.

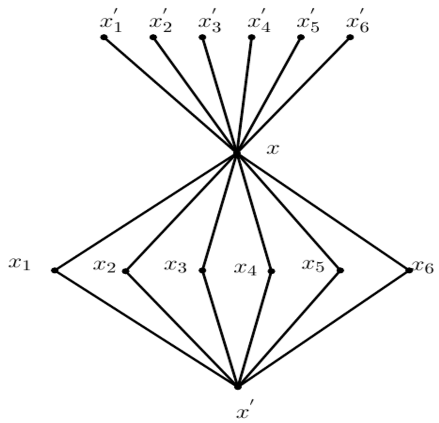


Figure: 3.9: $S'\left(K_{1,6}\right)$

Theorem 3.8

For the bistar graph $B_{m,n},(m,n\ge 1)$ $\gamma_{cer}^{c}\left(S'\left(B_{m,n}\right)\right)=2$.

Proof:

Consider the bistar graph $B_{m,n}$ with $m,n\ge 1$. Let x and y be the central vertices of $B_{m,n}$ and let $x_1,x_2,\dots x_m$ and $y_1,y_2,\dots,y_n$ be the end vertices adjacent to x and y in $B_{m,n}$, respectively. So $V\left(B_{m,n}\right)=\{x,y,x_i,y_j,/1\le i\le m,1\le j\le n\}$. Let $x',x_1',x_2',\dots,x_m',y',y_1',y_2',\dots,y_n'$ be the corresponding duplicated vertices of $x,x_1,x_2,\dots,x_m,y,y_1,y_2,\dots,y_n$ respectively which are added to obtained the graph $S'(B_{m,n})$ and is given in Figure 3.10. Then $V\left(S'\left(B_{m,n}\right)\right)=\{x,y,x_i,y_j,x',y',x_i',y_j'/1\le i\le m,1\le j\le n\}$ and so $\left|V\left(S'\left(B_{m,n}\right)\right)\right|=2m+2n+2$.

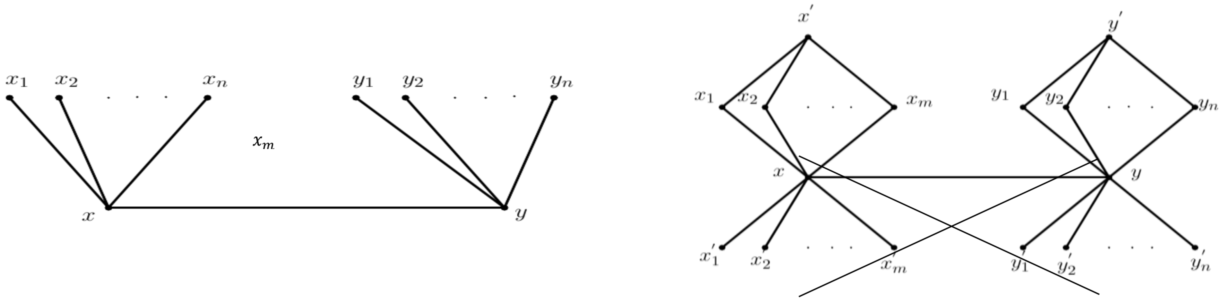


Figure 3.10: $B_{m,n}$, $S'(B_{m,n})$

Since x and y are the support vertices of $B_{m,n}$, by Theorem 3.5, $S = \{x, y\}$ is a subset of minimum connected certified dominating set of $S'(B_{m,n})$. Here x dominated the vertices x_i, x_i' and y'. Also y dominates the vertices y_j, y_j' and x'. Thus S is a dominating set of $S'(B_{m,n})$. Since $m, n \geq 1$, and x and y has at least two neighbours x_1, x_1' and y_1, y_1', respectivelyin $V\left(S'(B_{m,n})\right)\backslash S$. Thus that S is a certified dominating set of $S'(B_{m,n})$. Moreover, x and y are adjacent in $S'(B_{m,n})$, that S itself is a connected certified dominating set of $S(B_{m,n})$ and is $\gamma_{cer}^{c}\left(S'(B_{m,n})\right) \leq |S| = 2$. Furthermore, of we remove a vertex from the set S, it will not be a connected certified dominating set of $S'(B_{m,n})$. So, $\gamma_{cer}^{c}\left(S'(B_{m,n})\right) \leq |S| = 2$.Hence, $\gamma_{cer}^{c}\left(S'(B_{m,n})\right) = 2$.

Example 3.9

Consider the graph $S'(B_{3,4})$ given in Figure 3.11

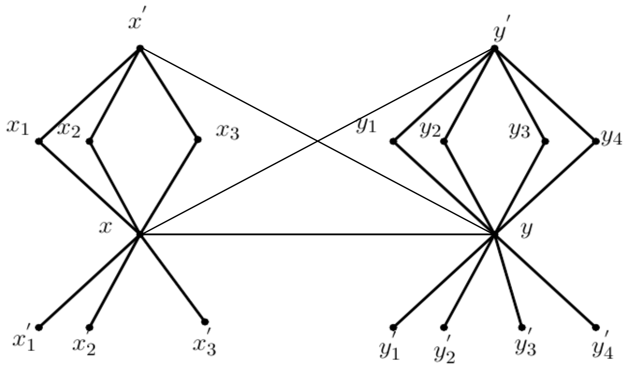


Figure 3.11: $S'(B_{3,4})$

By Theorem 3.8, $S = \{x, y\}$ is a minimum connected certified dominating set of $S'(B_{3,4})$ and here $\gamma_{cer}^{c}\left(S'(B_{3,4})\right) = 2$.

Theorem 3.10

For the complete bipartite graph, $K_{m,n}$ $(m, n \geq 2)$, $\gamma_{cer}^{c}\left(S'(K_{m,n})\right) = 2$.

Proof:

Consider the graph $K_{m,n}$ with vertex set $V(K_{m,n}) = \{x_i, y_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. Here $V_1 = \{x_i; 1 \leq i \leq m\}$ and $V_2 = \{y_j; 1 \leq j \leq n\}$ be the partition of $K_{m,n}$. So each x_i are the vertices adjacent with y_j and each y_j are the vertices adjacent with x_i. Let $x_i', 1 \leq i \leq m$ be the corresponding duplicated vertices of $x_i, 1 \leq i \leq m$ and $y'_j, 1 \leq j \leq n$ be the corresponding duplicated vertices of $y_j, 1 \leq j \leq n$, respectively in order to obtain $S'(K_{m,n})$ and is given in Figure 3.12. Then $V(S'(K_{m,n})) = \{x_i, y_j, x'_i, y'_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. So $\left|V\left(S'(K_{m,n})\right)\right| = 2(m+n)$.

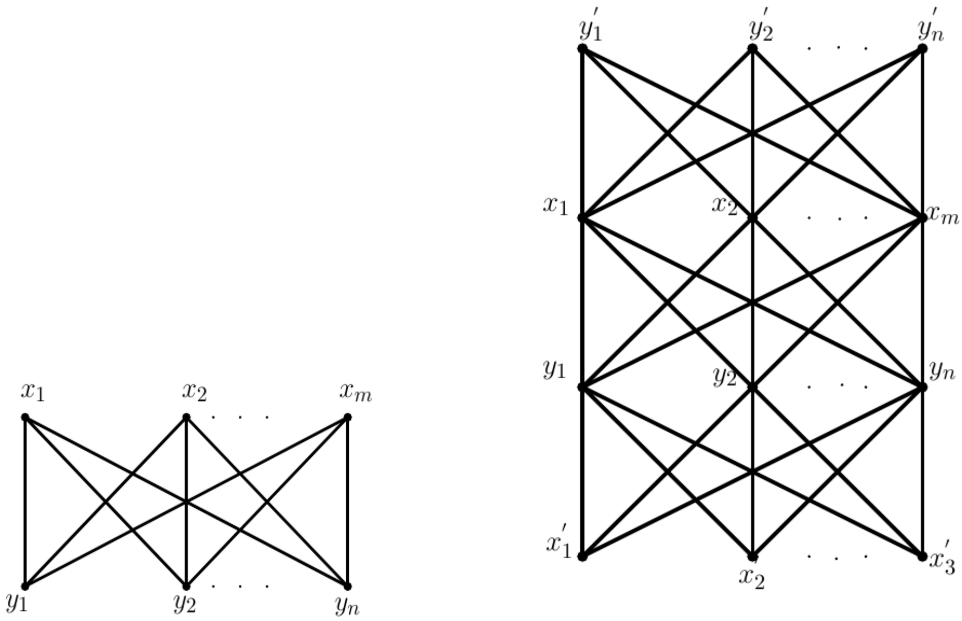


Figure 3.12: $K_{m,n}, S'(K_{m,n})$

In $S'(K_{m,n})$, for $1 \leq j \leq n, 1 \leq i \leq m, x_i$ is adjacent to every y_j and y'_j; y_j is adjacent to every x_i and $x'_i; x'_i$ is adjacent with every $y_j and\, y'_j$ is adjacent with every x_i. So clearly x_1 and y_1 dominates every vertices of $S'(K_{m,n})$. Thus, $S = \{x_1, y_1\}$ is a dominating set of $S'(K_{m,n})$. Since $m, n \geq 2$, every vertices in S has more that two neighbours in $V\left(S'(K_{m,n})\right) \backslash S$. Therefore that S is a certified dominating set of $S'(K_{m,n})$. Also, x_1 and y_1 are adjacent in $S'(K_{m,n})$. Hence, S is a connected certified dominating set for $S'(K_{m,n})$. Next we prove that S is of minimum cardinality with this property. If possible, let S_1 be any connected certified dominating set with $|S_1| < |S|$. Then there exists a vertex $v \in S_1$ with $v \notin S$. Consider the partitions of $S'(K_{m,n})$ by $X = \{x_1, x_2, \dots x_m, x'_1, x'_2, \dots, x'_m\}$ and $Y = \{y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n\}$. If $v \in S_1 \subseteq Y$, then some vertex of X is not dominated by S_1. Similarly, if $x \in S_1 \subseteq X$, then some vertex of Y is not dominated by the S_1. So S_1 is not a connected certified dominating set of $S'(K_{m,n})$. Hence, S is a minimum connected certified dominating set of $S'(K_{m,n})$ and hence $\gamma_{cer}^{c}\left(S'(K_{m,n})\right) = 2$.

Example 3.11

Consider the graph $S'(K_{5,5})$ given in Figure 3.13, $S = \{x_1, y_1\}$ is a minimum connected certified dominating set of $S'(K_{5,5})$ and hence $\gamma_{cer}^{c}\left(S'(K_{5,5})\right) = |S| = 2$.

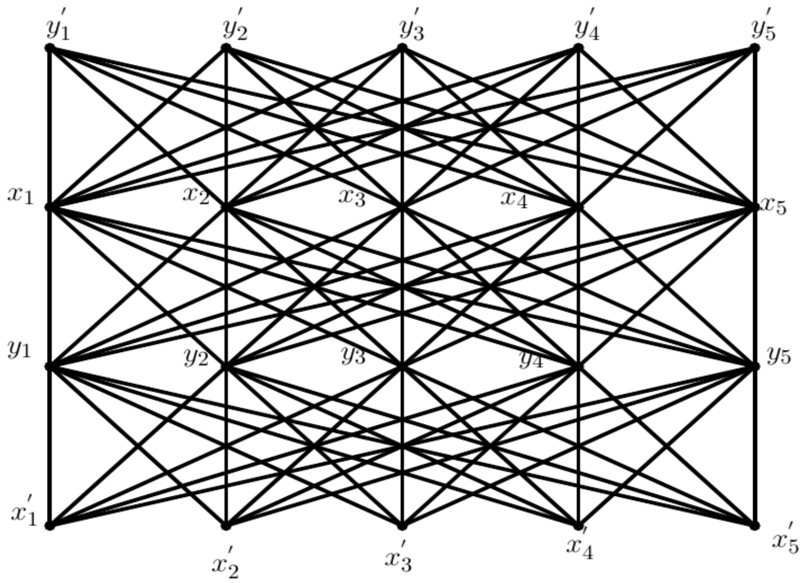


Figure 3.13: $S'(K_{5,5})$

Remark 3.12

For $m = n = 1, K_{m,n} = K_{1,1} \cong P_2 \cong K_2$, where $\gamma_{cer}^c(K_{1,1}) = 2$ and $S'(K_{1,1}) \cong P_4$, then $\gamma_{cer}^c\left(S'(K_{1,1})\right) = 4$.

In this case, $\gamma_{cer}^c\left(S'(K_{n,n})\right) > \gamma_{cer}^c(K_{n,n})$.

For $m = 2, n \geq 3$ *or* $n = 2, m \geq 3, K_{m,n} = K_{2,n}$. So by Observation 2.3, $\gamma_{cer}^c(K_{m,n}) = 3$ and also by Theorem 3.10, $\gamma_{cer}^c\left(S'(K_{m,n})\right) = 2$.

In this case, $\gamma_{cer}^c\left(S'(K_{m,n})\right) < \gamma_{cer}^c(K_{m,n})$.

For $m, n \geq 3, \gamma_{cer}^c(K_{m,n}) = \gamma_{cer}^c\left(S'(K_{m,n})\right)$. This follows from Observation 2.3 and Theorem 3.12

Theorem 3.13

For positive integer $n \geq 2$, $\gamma_{cer}^c(S'(P_n)) = \begin{cases} 4 \text{ if } n = 2,3 \\ n-2 \text{ if } n \geq 4 \end{cases}$.

Proof

Let $x_1, x_2, \dots, x_n$ be the vertices of the path P_n which are duplicated by the vertices $x_1', x_2', \dots, x_n'$ respectively to obtain $S'(P_n)$ and is given in Figure 3.14. Then, $V(S'(P_n)) = \{x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n'\}$ and so $|V(S'(P_n)| = 2n$.

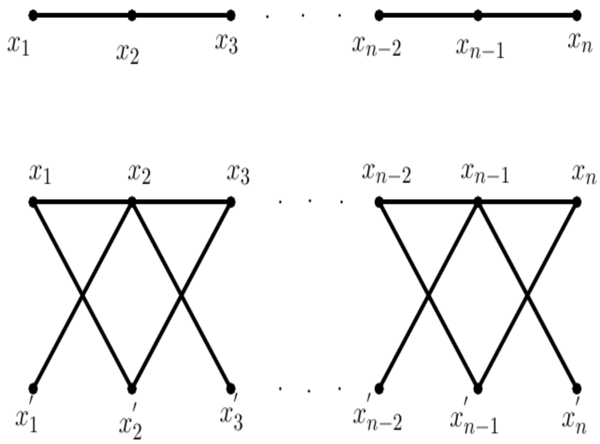


Figure 3.14: $P_n, S'(P_n)$

If $n = 2$, then $S'(P_2) \cong P_4$ and so by Observation 2.4, $\gamma_{cer}^{c}(S'(P_2)) = 4$.

For $n = 3, x_2$ is the only support vertex of P_3. Therefore by Theorem 3.5, $\{x_2\}$ be in every connected certified dominating set of $S'(P_3)$. Here $N[x_2] = \{x_1, x_2, x_3, x_1', x_3'\}$. Therefore, S dominates $V(S'(P_3))\{x_2\}$. so it is clear that $S - \{x_2, x_2'\}$ is a dominating set of $S'(P_2)$. Also x_2 has four neighbors in $V(S'(P_3))\backslash S$. So that S is a certified dominating set of $S'(P_3)$. But that x_2 and x_2' are not connected. Thus, that S is not a connected certified dominating set of $S'(P_3)$. Since x_2 and x_2' are connected by two paths through x_1 *and* x_3, respectively of distance 2. If we select $S_1 = S \cup \{x_1\}$ or $S_1 = S \cup \{x_3\}$ then x_2' has exactly one neighbor in $V(S'(P_3))\backslash S_1$, which is not possible. Also, if we select $S_1 = S \cup \{x_1'\}$ or $S_1 = S \cup \{x_3'\}$, then S_1 is not a connected certified dominating set of $S'(P_3)$. Thus $\gamma_{cer}^{c}(S'(P_3)) > |S_1| = 4$. Now, if we select $S_1 = S \cup \{x_2, x_3\}$, then S_1 forms a connected certified dominating set of $S'(P_3)$ and so $\gamma_{cer}^{c}(S'(P_3)) \leq |S_1| = 4$.\

Hence $\gamma_{cer}^{c}(S'(P_3)) = 4$.

Now consider $n \geq 4$. Here x_2 and x_{n-1} are the support vertices of P_n. So by Theorem 3.5, $S = \{x_2, x_{n-1}\}$ is a subset of every connected certified dominating set of $S'(P_n)$. Let S_1 be a minimum connected certified dominating set of $S'(P_n)$. Since x_2' and x_{n-1}' are not dominated by any vertex of S, that S itself is not a connected certified dominating set of $S'(P_n)$. So, $S \subset S_1$. Now we need to construct that S' as a connected certified dominating set of $S'(P_n)$. By the definition of connectedness, to dominate x_2', we must select either x_1 or x_3. If we select $x_1 \in S_1$, then s_1 has exactly one neighbor x_2' in $V(S'(P_n))\backslash S_1$. So x_3 must be in S_1. Similarly to dominate x_{n-1}'. We must select x_{n-2} in S_1. Now, clearly x_2 dominates x_1' and x_3'. Also x_{n-1} dominates x_n' and x_{n-2}'. x_4' dominated by x_3. To dominate x_5', we select x_4 in S_1. Since v_2 and v_{n-1} are connected by a unique path through the vertices a $x_3, x_4, \ldots, x_{n-1}\, x_{n-2}$ in P_n and that $< S_1 >$ is connected. So $x_3, x_4, \ldots, x_{n-1} \in S_1$. If $\{x_2, x_3, \ldots, x_{n-1}\} = S_1$, then S_1 form a connected certified dominating set of $S'(P_n)$ and so $\gamma_{cer}^{c}(S'(P_n)) \leq |S_1| = n - 2$. Furthermore if we remove any vertex from $\{x_2, x_3, \ldots x_{n-1}\} = S_1$, then S_1 is not a connected certified dominating set of $S'(P_n)$. Therefore S_1 itself is a minimum connected certified dominating set of $S'(P_n)$ and hence, $\gamma_{cer}^{c}(S'(P_n)) = n - 2$.

Example 3.14

Consider $S'(P_6)$ given in Figure 3.15

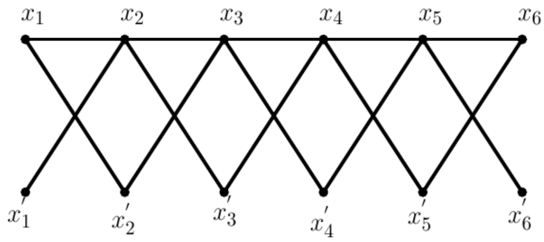


Figure 3.15: $S'(P_6)$

By Theorem 3.13, $S = \{x_2, x_3, x_4, x_5\}$ is a minimum connected certified dominating set of $S'(P_6)$ and so $\gamma_{cer}^{c}(S'(P_6)) = |S| = 4$.

Theorem 3.15

For positive integer $n \geq 3, \gamma_{cer}^{c}(S'(C_n)) = \begin{cases} n-1 \; if \; n = 3 \\ n-2 \; if \; n \geq 4 \end{cases}$.

Proof

Let $x_1 x_2 \dots x_n$ be the cycle C_n and let $x_1', x_2', \dots, x_n'$ be the corresponding duplicated vertices of $x_1, x_2, \dots, x_n$ respectively added to obtain $S'(C_n)$ and the graph is given in Figure 3.16

Then $V(S'(C_n)) = \{x_1, x_2, \dots x_n, x_1', x_2', \dots, x_n'\}$ and so $|V(S'(C_n))| = 2n$. Also, $S'(C_n)$ has no support vertices.

If $n = 3$, then $N[x_1] = \{x_1, x_2, x_3, x_2', x_3'\}$ $N[x_2] = \{x_1, x_2, x_3, x_1', x_3'\}$,

$N[x_3] = \{x_1, x_2, x_3, x_2', x_1'\}, N[x_1'] = \{x_1', x_2, x_3\}, N[x_2'] = \{x_2', x_1, x_3\}$, and

$N[x_3'] = \{x_1, x_2, x_3'\}$. Clearly, $N[x_1] \cup N[x_2] = V(S'(C_n))$. Hence, we obtain a dominating set $S = \{x_1, x_2\}$ of $S'(C_n)$. Also, x_1 has three neighbors in $V(S'(C_n))\backslash S$ and x_2 has three neighbors in $V(S'(C_n))\backslash S$. Therefore that S is a certified dominating set of $S'(C_n)$. Since x_1 and x_2 are adjacent in $S'(C_n)$ that S itself is a connected certified dominating set of $S'(C_n)$ and so $\gamma_{cer}^{c}(S'(C_n)) \leq |S| = 2$. So no vertex is adjacent with all the vertices of $S'(C_n)$, we have, $\gamma_{cer}^{c}(S'(C_n)) = 2 = n - 1$.

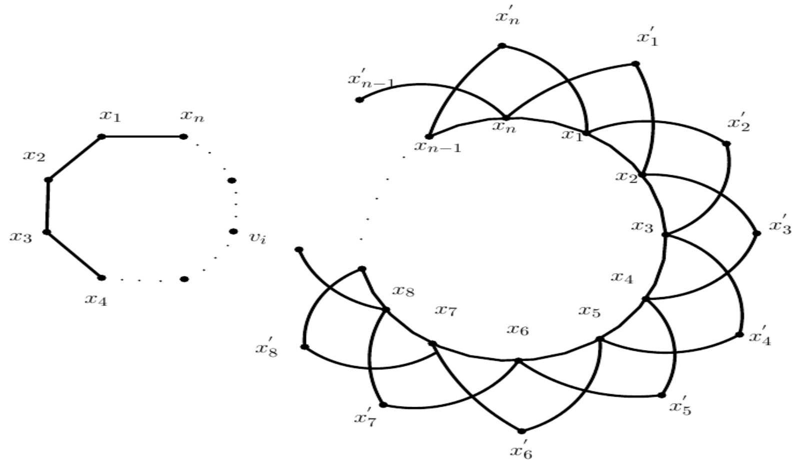


Figure 3.16: $C_n, S'(C_n)$

Now assume $n \geq 4$.

In this, we select $S = \{x_1, x_2, \dots, x_{n-2}\}$. Clearly $N[S] = V(S'(C_n))$ and every vertices in S has more than two neighbors in $V(S'(C_n))\backslash S$. Also $< S >$ is connected so that is a connected certified dominating set of $S'(C_n)$. Moreover, if we remove any vertex from that S is not a connected certified dominating set of $S'(C_n)$. Therefore, that S is a minimal connected certified dominating of $S'(C_n)$. Furthermore, there does not exists a connected certified dominating set of cardinality less than S. Thus S is a minimum connected certified dominating set of $S'(C_n)$ and hence, $\gamma_{cer}^{c}(S'(C_n)) = n - 2$.

Therefore $\gamma_{cer}^{c}(S'(C_n)) = \begin{cases} n-1 \text{ } if \text{ } n = 3 \\ n-2 \text{ } if \text{ } n \geq 4 \end{cases}$.

Example 3.16

Consider $S'(C_{12})$ given in Figure 3.17. By Theorem 3.15, $S' = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ is a minimum connected certified dominating set of $S'(C_{12})$ and hence $\gamma_{cer}^{c}(S'(C_{12})) = |S| = 10$. Now consider the graph $S'(C_3)$ given in Figure 3.17. Here by Theorem 3.15 $S = \{x_1, x_2\}$ is a minimum connected certified dominating set of $S'(C_3)$ and hence $\gamma_{cer}^{c}(S'(C_3)) = |S| = 2$.

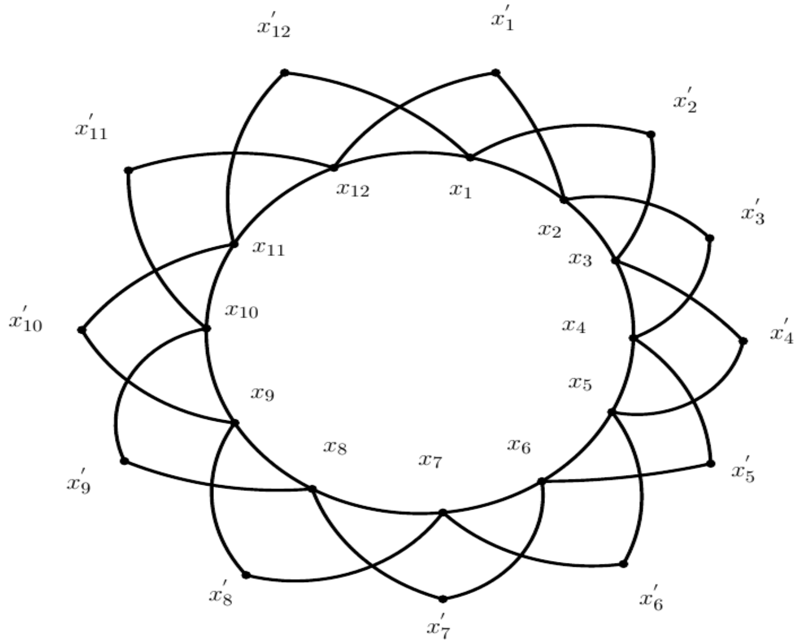


Figure 3.17: $S'(C_{12})$

Theorem 3.17

For the wheel graph $W_n, n \geq 4$ $\gamma_{cer}^{c}(S'(W_n)) = 2$.

Proof

Let $x_1, x_2, \dots, x_{n-1}$ be the rim vertices of W_n and x is the apex vertex of W_n. Let $x_1', x_2', \dots, x_{n-1}'$ be the corresponding duplicated vertices of $x_1, x_2, \dots, x_{n-1}$ respectively and x' be the duplicated vertex of x which are added to obtain $S'(W_n)$. So $V(S'(W_n)) = \{x_1, x_2, \dots, x_{n-1}, x_1', x_2', \dots, x_{n-1}'\}$ and the number of vertices in $S'(W_n)$ is $2n$. This graph is in Figure 3.18

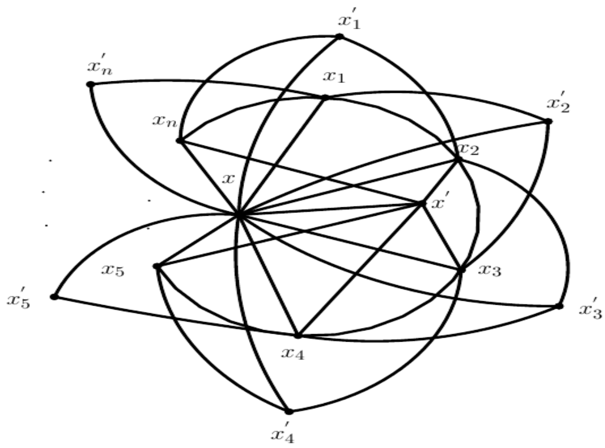


Figure 3.18: $S'(W_n)$

Clearly $S'(W_n)$ has no support vertices and no full degree vertices. Therefore by Theorem $\gamma_{cer}^{c}(S'(W_n)) \geq 2$. Here x is adjacent to every vertices other that x' and x' is adjacent to every x_i for $1 \leq x \leq n$. Now we construct a connected certified dominating set S of $S'(W_n)$ with cardinality two. If we consider $S = \{x, x'\}$, then $N[x] \cup N[x'] = V(S'(W_n))$. So S is a dominating set of $S'(W_n)$. Also every vertices in S has at least two neighbours in $V(S'(W_n)) \backslash S$. Therefore, that S is a certified dominating set of $S'(W_n)$. But x is not adjacent with x'. So $< S >$ is not connected. Thus $S = \{x, x'\}$ is not a connected certified dominating set of $S'(W_n)$. Since x is adjacent to every vertices other that x' and x' is adjacent to every $x_i, 1 \leq i \leq n-1, S = \{x', x_i\}, 1 \leq i \leq n-1$ is a certified dominating set of $S'(W_n)$. Also, $< S >$ is connected. Therefore $S = \{x', x_i\}$ is a connected certified dominating set of $S'(W_n)$ and so, $\gamma_{cer}^{c}(S'(W_n)) \leq |S| = 2$. Hence $\gamma_{cer}^{c}(S'(W_n)) = 2$.

Example 3.18

Consider $S'(W_7)$ given in Figure 3.19. By Theorem 3.17, $S = \{x, x_1\}$ is a minimum connected certified dominating set of $S'(W_7)$ and hence, $\gamma_{cer}^{c}(S'(W_7)) = |S| = 2$.

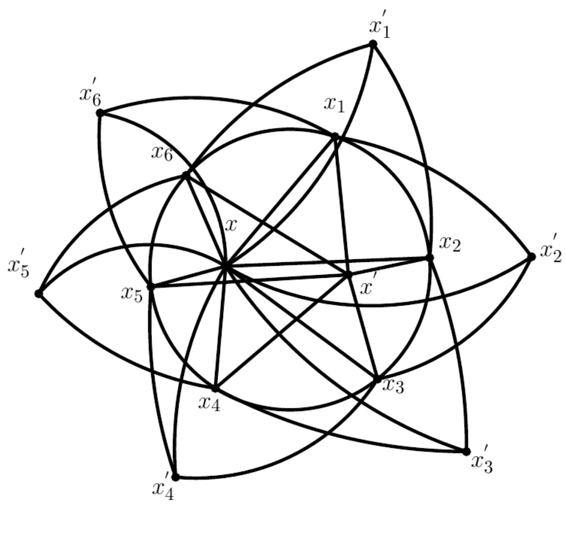


Figure 3.19: $S'(W_7)$

Theorem 3.19

For any integer$n \geq 4, \gamma_{cer}^{c}\left(S'(F_n)\right) = 2$.

Proof

Let $x_1, x_2, \dots, x_{n-1}$ be the n-vertices of F_n, where x is the apex vertex of F_n. Let $x_1', x_2', \dots, x_{n-1}'$ be the corresponding duplicated vertices of $x_1, x_2 \dots, x_{n-1}$ respectively and x' be the duplicated vertex of x which are added to obtain $S'(F_n)$ and is given in Figure 3.20

Here $V\left(S'(F_n)\right) = \{x_1, x_2, \dots, x_{n-1}, x_1', x_2', \dots, x_{n-1}'\}$ and so $|V\left(S'(F_n)\right) = 2n$. Since $S'(F_n)$ has no full degree vertices, $\gamma_{cer}^{c}\left(S'(F_n)\right) \geq 2$.

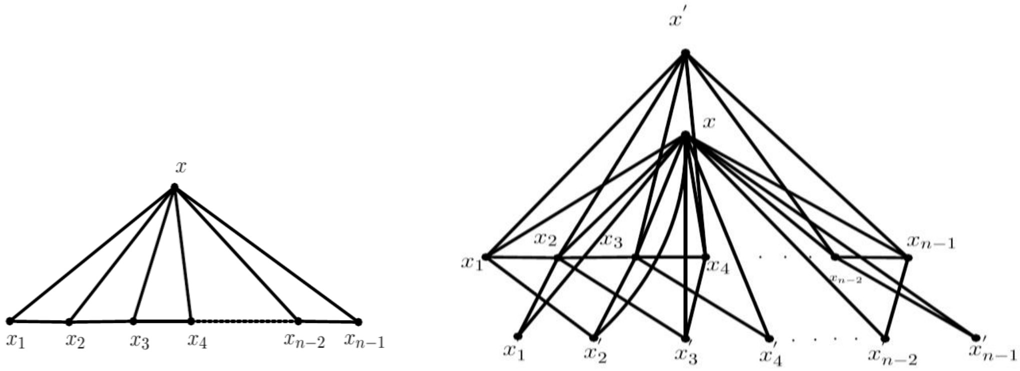


Figure 3.20: $F_n, S'(F_n)$

Now we construct a connected certified dominating set S of $S'(F_n)$ with cardinality two. As in theorem 3.15, similar if we consider $S = \{x, x'\}$ is a certified dominating set of $S'(F_n)$. But the subgraph induced by S is not a connected certified dominating set of $S'(F_n)$ since x is adjacent with every vertices of $S'(F_n)\{x'\}$, that x dominates every vertices $S'(F_n)\backslash\{x'\}$. To dominate x' we must select a vertex from a vertex of F_n that is if $S_i = \{x, x_i\}$ for $1 \leq i \leq n-1$ is a connected certified dominating set of $S'(F_n)$ and so $\gamma_{cer}^{c}\left(S'(F_n)\right) \leq |S_i| = 2$. Hence, we conclude, $\gamma_{cer}^{c}\left(S'(F_n)\right) = 2$.

Example 3.20

Consider $S'(F_7)$ given in Figure 3.21. By Theorem 3.19, $S = \{x, x_1\}$ is a minimum connected certified dominating set of $S'(F_7)$ and so $\gamma_{cer}^{c}\left(S'(F_n)\right) = |S| = 2$.

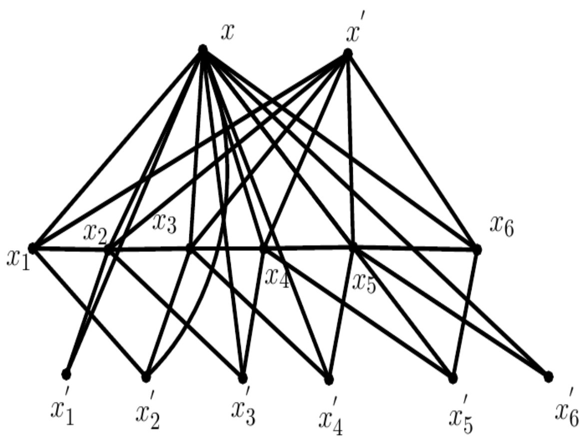


Figure 3.21: $S'(F_7)$

Theorem 3.21

For the helm graph $H_n (n \geq 4), \gamma_{cer}^c(S'(H_n)) = n - 1$.

Proof

For $n \geq 4$, let H_n be the helm graph derived from a wheel graph W_n by attaching a pendent edge them with each rim vertices $x_1, x_2, \dots, x_{n-1}$. Here in W_n, x is the apex vertex and $y_1, y_2, \dots, y_{n-1}$ be the pendent vertices. Let $x'_1, x'_2, \dots, x'_{n-1}$ be the duplicated vertices of $x_1, x_2, \dots, x_{n-1}$ respectively and let $y'_1, y'_2, \dots, y'_{n-1}$ be the duplicated vertices of $y_1, y_2, \dots, y_{n-1}$ respectively, and x' be the duplicated vertex of x to obtain the graph $S'(H_n)$. Then $V(S'(H_n)) = \{x_1, x_2, \dots, x_{n-1}, y_1, y_2, \dots, y_{n-1}, x'_1, x'_2, \dots, x'_{n-1}, y'_1, y'_2, \dots, y'_{n-1}, x'\}$ and so $|V(S'(H_n))| = 4n - 2$. Thus the graph given in Figure 3.22

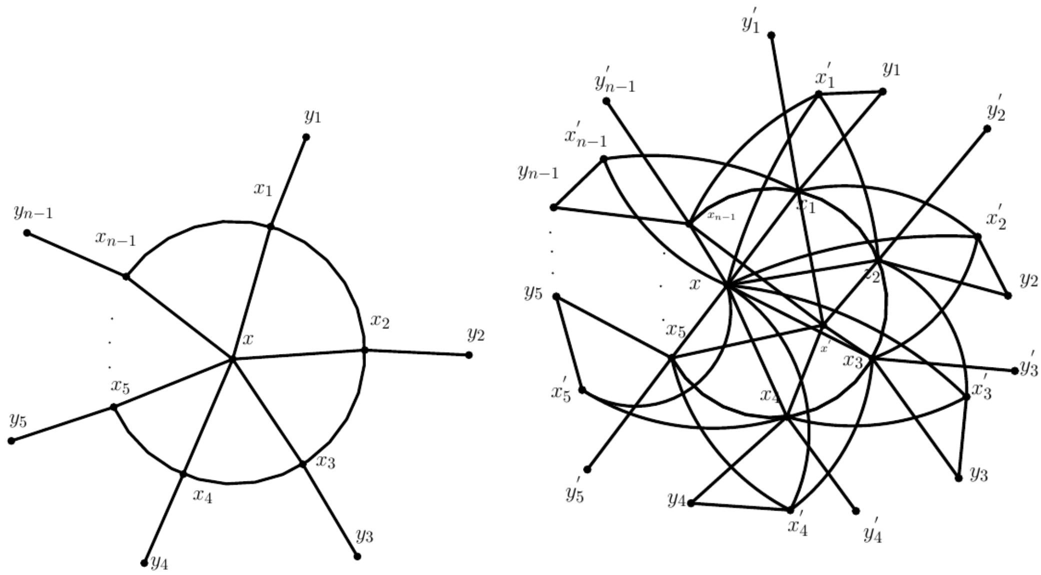


Figure 3.22: $H_n, S'(H_n)$

Let $S = \{x_1, x_2, \dots, x_{n-1}\}$ be the set of all support vertices of H_n. By Theorem 3.5, $\gamma_{cer}^c(S'(H_n)) \geq n - 1$. Since x_i is adjacent with y_i, y'_i, x and x', that x_i dominates every vertices of $S'(H_n)$ for $1 \leq i \leq n - 1$, that S itself dominates every vertices in $S'(H_n)$. Since y_i and y'_i must adjacent to x_i for $1 \leq i \leq n - 1$, every vertices in S has at least two neighbor in $V(S'(H_n))\backslash S$. So S is a certified dominating set of $S'(H_n)$. Moreover each v_i are the rim vertices of W_n, the subgraph included by S is connected. Therefore S itself is a connected certified dominating set of $S'(H_n)$ and so $\gamma_{cer}^c(S'(H_n)) \leq |S| = n - 1$. Hence, we concludes, $\gamma_{cer}^c(S'(H_n)) = n - 1$.

Example 3.22

Consider $S'(H_7)$ give in Figure 3.23. By Theorem 3.21, $S = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a minimum connected certified dominating set of $S'(H_7)$. Hence $\gamma_{cer}^c(S'(H_7))$.

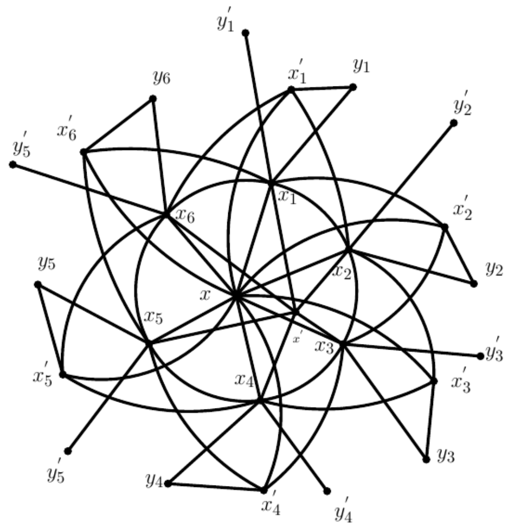


Figure 3.23: $S'(H_7)$

References

[1] J.A. Bondy and U.S.R. Murty, *Graph theory*, Graduate texts in mathematics, vol. 244, springer Science and Meda, 2008.

[2] T.W.Haynes, S.T. Hedetneimi and P.J.Slater, *Fundamental of Domination in Graphs*, Marcel Dekker, New York, 1998.

[3] T.W.Haynes, S.T. Hedetneimi and P.J.Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[4] E. Sampath Kumar, H.B. Walikar On the Splitting Graph of a Graph, The Karnataka University Journal, Science-Vol. XXV and XXVI(Combined) (1980-1981).

[5] C. Jayasekaran and S.V. Ashwin Prakash, *Detour Global Domination for Splitting Graph of Path, Cycle, Wheel and Helm graph* , Mathematical Statistician and Engineering Applications, 72(1), 185-194(2023).

[6] C. Jayasekaran and S.V. Ashwin Prakash, Detour Global Domination for Splitting graph, Discrete Mathematics Algorithms and Applications 15(05) (2022).

[7] S. Durai Raj, S.G. Shiji Kumari and A.M.Anto, Certified Domination Number in product of Graphs, *Turkish Journal of Computer and Mathematical Education* 11(3), 1166-1170(2020).

[8] S. Durai Raj, S.G. Shiji Kumari and A.M.Anto, Certified Domination Number in Corona Product of Graphs, Malaya Journal of Mathematik 9(1), 1080-1082 (2021).

[9] M. Dettlaft, M. Lemansko, J. Topp, R. Ziemann and P. Zylinski, Certified Domination, AKCE *International Journal of Graphs and Combinatorics* (Article press), (2018).

[10] Azham IIyass, Vishwajeet S, Goswami, Connected certified Domination Number of certain graphs, Advancements in Physical & Mathematical Sciences, AP Conference Proceedings, 2735(1) (2023).