

Efficient Techniques for Solving System of (2+1) D & (3+1) D PDES Using Rangaig Transforms Based HAM

¹Inderdeep Singh, ²Sandeep Sharma

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Abstract: In this study, we explore the application of Rangaig transforms combined with the Homotopy Analysis Method (HAM) for solving systems of (2+1)D and (3+1)D nonlinear partial differential equations. Nonlinear PDEs frequently arise in various scientific and engineering fields, modeling complex phenomena such as fluid dynamics, heat transfer, and wave propagation. Traditional numerical and analytical methods often encounter limitations when addressing such systems, particularly in higher dimensions. The Rangaig transform, a powerful integral transform, is integrated with HAM to efficiently construct approximate analytical solutions to these complex systems. The proposed methodology leverages the strengths of both techniques: the Rangaig transform simplifies the PDEs by converting them into a more tractable form, while HAM provides a flexible framework for obtaining convergent series solutions without restrictive assumptions on small parameters. We demonstrate the effectiveness of this combined approach through several benchmark problems, showcasing its accuracy, efficiency, and convergence properties. The results indicate that the Rangaig-HAM hybrid approach not only accelerates the solution process but also extends the applicability of HAM to more challenging nonlinear PDE systems in multidimensional spaces. This study contributes to advancing analytical techniques for solving high-dimensional nonlinear PDEs and provides a robust tool for researchers and engineers dealing with complex physical systems.

Keywords: “Rangaig Transform”, “Homotopy Analysis Method”, (2+1)-D and (3+1)-D system, PDEs, Test Examples.

1 Introduction

The Nonlinear “partial differential equations (PDEs)” form a basis for the description of various physical and engineering processes, such as fluid dynamics, heat transfer, and propagation of electromagnetic waves. However, solving such PDEs unfortunately remains a formidable task especially in two and three dimensions because of the underlying complexity and nonlinearity.

Optimal solutions are difficult to obtain, and though numerical methods can be employed they are quite often computational and numerical in higher dimensions.

In response to these difficulties, some approaches have been searched for compound numerical methods that use integral transforms with effective approximations. Among these, the Rangaig transform has attracted much attention if needed for converting differential equations into algebraic forms. By contrast, the “Homotopy Analysis Method (HAM)”, developed by Liao, is a powerful tool to obtain the series solutions of the nonlinear problems without making the perturbative assumptions.

Liao introduced the “Homotopy Analysis Method (HAM)” in 1992 for his PhD dissertation [12]-[16]. Recent developments in this technique involve combining it with

¹ Associate Professor, Department of Physical Science, Sant Baba Bhag Singh University, Jalandhar-144030, Punjab, India

Email: ¹inderdeeps.ma.12@gmail.com

² Research Scholar, Department of Physical Science, Sant Baba Bhag Singh University, Jalandhar-144030, Punjab, India

Email: ²sandeepsharma200@gmail.com

Corresponding Author: Sandeep Sharma

various transform methods, including Elzaki, [4] Laplace [6], Sumudu [3], Aboodh [1], and Natural transform method [9], Rangaig [20]. transforms. These integrations have resulted in new approaches such as “The Laplace transform with homotopy analysis method” [24], “The Shehu transform with homotopy analysis” [16], “The Sumudu transform with homotopy analysis” [21], the “Elzaki transform with homotopy analysis” [4], [5].

Looking at the “Homotopy Perturbation Method (HPM)”, “Adomian Decomposition Method (ADM)”, “Variational Iteration Method (VIM)”, “Differential Transform Method (DTM)” and the “Laplace Adomian Decomposition Method (LADM)”, one sees some problems that limit their effectiveness in solving complex nonlinear problems. Due to the fact that HPM involves heuristic selection of parameters, its major disadvantage is lack of reliability and complexity, when managing many solutions. It has been found that is ADMs problem solving capabilities are restricted by computational problems inherent with complex polynomial recursion and confusion regarding convergence, particularly in isolation. VIM itself has problems of solving the optimum Lagrange multiplier and has a high dependency on the first approximations and is not capable of providing tightly bound estimates for non-smooth approximate solutions. DTM performs worse as dimensions or significant nonlinearities of the system increase; conversely, the practical application of LADM has difficulties with the inversion of Laplace transforms and consideration of boundary value issues. Common constraints of these approaches are convergence uncertainty, parameter dependency, heightened computing complexity, restricted application to singularities and discontinuities, and the necessity for substantial manual intervention. Resolving these obstacles necessitates hybrid methodology that include the advantages of current techniques, sophisticated convergence frameworks, and novel strategies for high-dimensional and single issues. This research highlights the need for advanced or hybrid

models to address the ongoing constraints of existing approaches.

In this paper, we integrate the Rangaig transform with HAM to create a versatile tool for solving multidimensional nonlinear PDEs. The combined methodology leverages the advantages of both techniques, enabling efficient simplification and accurate solution construction. Several case studies are presented to demonstrate the efficacy of this approach, highlighting its potential for tackling real-world problems.

The conceptual framework of this research study is as follows: The Rangaig transform's fundamental definitions and ideas are covered in Section 2. In Section 3, the homotopy analysis technique was covered. In Section 4, test experiments have been conducted to solve based on a mix of Homotopy analysis and the Rangaig transform methodology of the “nonlinear partial differential equations” system in (2+1)-D and (3+1)-D. In Section 5, the conclusion is covered.

2 Concept of Rangaig Transform [18], [20], [21], [25]

This segment presents the foundational definitions and characteristics of the Rangaig Transform, which are essential for the discussions in this paper. The Rangaig transform represents a recent mathematical technique devised for functions with exponential growth. Our attention is directed towards operations in the set $\mathcal{H}$, as outlined in reference [25].

$$\mathcal{H} = \{h(t) \exists \mathcal{N}, \mathcal{A}_1, \mathcal{A}_2 > 0, |h(t)| > \mathcal{N}e^{\mathcal{A}_1|t|}, t \in (-1)^{i-1} \times (-\infty, 0)\} \quad (1)$$

$\mathcal{N}$, the arbitrary constant. The arbitrary constants $\mathcal{A}_1$ and $\mathcal{A}_2$ can possess either an infinite or a perpetually finite value. We now

present a novel transformation that may be included in (I) as follows:

$$\begin{aligned}\Re[\mathcal{H}(t)] &= \mathcal{T}(\rho) \\ &= \frac{1}{\rho} \int_{-\infty}^0 e^{\rho t} \mathcal{H}(t) dt, \frac{1}{\mathcal{A}_1} \leq \rho \leq \frac{1}{\mathcal{A}_2}\end{aligned}\quad (2)$$

The definition can be stated as a transformation where the factor ρ substitutes for the variable t within the function $\mathcal{H}$, known as the Rangaig Transform. On the other hand, one can state that there is transitioning, in terms of describing the function $\mathcal{H}(t)$ in the rho space $\mathcal{T}(\rho)$. The following exercise exemplify how the “Rangaig Transform” can be used in order to arrive at solutions regarding a certain type of function. [18], [20], [21], [25]

The subsequent “Rangaig Transform” is attained in some core forms [18]. The general function is: - $\eta(t) = \Re\{\eta(t)\}$

- $\Re\{\eta(t)\} = \Re\{1\} = \frac{1}{\rho^2}$
- $\Re\{1\} = -\frac{1}{\rho^3}$
- $\Re\{t\} = -\frac{1}{\rho^3}$
- $\Re\{t^n, n \geq 0\} = \frac{(-1)^2 n!}{\rho^{n+2}}$
- $\Re\{\sin(t)\} = -\frac{1}{\rho(\rho^2+1)}$
- $\Re\{\cos(t)\} = \frac{1}{(\rho^2+1)}$
- $\Re\{t^n, n \leq 0\} = \frac{(-1)^{n+1} \Gamma(-n)}{\rho^n}$
- $\Re\{e^{at}\} = \frac{1}{\rho(\rho+a)}$
- $\Re\{M(t-a)\} = \frac{1}{\rho^2} e^{at}$

Theorem I: - [20] The “Transformation of Rangaig Derivatives”. If

$\mathcal{H}(t), \mathcal{H}^1(t), \mathcal{H}^n(t) \in \mathcal{H}$, then

$$\begin{aligned}\Re[\mathcal{H}^n(t)] &= \mathcal{T}(\rho) \\ &= (-1)^n \rho^n \mathcal{T}(\rho) + (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \rho^{n-2-k} \mathcal{H}^{(k)}(0)\end{aligned}\quad (3)$$

Theorem II: - [20] The “Transformation of Rangaig Integrals”. If.

$$m^n(t) = \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \dots \int_0^{t_{n+1}} \mathcal{H}(\tau) (d\tau)^n,$$

Where $m(t)$ belongs to $\mathcal{H}$. Afterwards, we define the Rangaig Transform of $m^n(t)$ as follows:

$$\Re[m^n(t)] = \mathcal{T}_n(\rho) = \left(\frac{-1}{\rho}\right)^n \mathcal{T}(\rho)$$

Theorem III: - [20] The convolution identity's Rangaig Transform is defined as:

$$\Re[(\mathcal{H} * \mathcal{G})(t)] = -\rho \mathcal{T}_1(\rho) \mathcal{T}_2(\rho)$$

Where $(\mathcal{H} * \mathcal{G})(t) = \int_0^t \mathcal{H}(t-\psi) \mathcal{G}(\psi) d\tau$, $\mathcal{T}_1(\rho)$ and $\mathcal{T}_2(\rho)$ are, correspondingly, the Rangaig transforms of $\mathcal{H}(t)$ and $\mathcal{G}(t)$.

Theorem IV: - [20] (Duality relation of R-Transform and L-Transform), The attributes of the Rangaig Transform $\mathcal{T}(\rho)$ in relation to "The Laplace Transform." $\mathcal{F}(\rho)$ of $\mathcal{H}(t)$ is defined as follows if (t) and $\mathcal{H}(-t)$ are present across $\mathcal{H}$.

$$\mathcal{T}(\rho) = \frac{1}{\rho} \mathcal{F}(-\rho)$$

Proposition-I: - If $\frac{\partial W(x,t)}{\partial t}$ exist, the following results can be obtained by using integration by parts:

$$\begin{aligned}\Re\left[\frac{\partial W(x,t)}{\partial t}\right] &= -\rho \mathcal{T}(x, \rho) \\ &\quad + \frac{1}{\rho} W(x, 0)\end{aligned}\quad (4)$$

Demonstration- We utilize formula (2) and the concept of integration of components to demonstrate this.

Proposition-II: - Assuming $\mathcal{T}(x, \rho)$ represents the “Rangaig Transform” of $\eta(x, t)$, the following is obtained:

$$\begin{aligned} & \Re \left[\frac{\partial^n W(\mathbf{x}, t)}{\partial t^n} \right] \\ &= (-1)^n \rho^n \mathcal{T}(\mathbf{x}, \rho) \\ &+ (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \rho^{n-2-k} \frac{\partial^k W(\mathbf{x}, 0)}{\partial t^k} \end{aligned} \quad (5)$$

Demonstration- We demonstrate that (5) is correct by use of mathematical induction. By assuming $n = 1$ and utilizing formula (5), we derive:

$$\Re \left[\frac{\partial W(\mathbf{x}, t)}{\partial t} \right] = -\rho \mathcal{T}(\mathbf{x}, \rho) + \frac{1}{\rho} W(\mathbf{x}, 0) \quad (6)$$

Therefore, using (4), we see that the formula is valid for $n = 1$. Make an inductive assumption that the formula is true for n , which means

$$\begin{aligned} & \Re \left[\frac{\partial^n W(\mathbf{x}, t)}{\partial t^n} \right] \\ &= (-1)^n \rho^n \mathcal{T}(\mathbf{x}, \rho) \\ &+ (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \rho^{n-2-k} \frac{\partial^k W(\mathbf{x}, 0)}{\partial t^k} \end{aligned} \quad (7)$$

Show that it is still true at rank $n + 1$. Assume $\frac{\partial^n W(\mathbf{x}, t)}{\partial t^n} = v(\mathbf{x}, t)$ and based on equations (4) and (7), we get:

$$\begin{aligned} &= \Re \left[\frac{\partial^{n+1} W(\mathbf{x}, t)}{\partial t^{n+1}} \right] = \Re \left[\frac{\partial v(\mathbf{x}, t)}{\partial t} \right] \\ &= -\rho \Re[v(\mathbf{x}, t)] + \frac{1}{\rho} v(\mathbf{x}, 0) \\ &= -\rho \left[(-1)^n \rho^n \mathcal{T}(\mathbf{x}, \rho) \right. \\ &+ (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \rho^{n-2-k} \frac{\partial^k W(\mathbf{x}, 0)}{\partial t^k} \left. \right] \\ &+ \frac{1}{\rho} \frac{\partial^n W(\mathbf{x}, t)}{\partial t^n} \end{aligned}$$

$$\begin{aligned} &= (-1)^{n+1} \rho^{n+1} \mathcal{T}(\mathbf{x}, \rho) + (-1)^{n+2} \sum_{k=0}^{n-1} (-1)^k \rho^{n-1-k} \frac{\partial^k W(\mathbf{x}, 0)}{\partial t^k} \\ &+ \frac{1}{\rho} \frac{\partial^n W(\mathbf{x}, t)}{\partial t^n} \\ &= (-1)^{n+1} \rho^{n+1} \mathcal{T}(\mathbf{x}, \rho) + (-1)^{n+2} \sum_{k=0}^n (-1)^k \rho^{n-1-k} \frac{\partial^k W(\mathbf{x}, 0)}{\partial t^k} \end{aligned}$$

Consequently, the mathematical induction principle states that the formula (5) is valid for every $n \geq 1$.

3 Introduction to Homotopy Analysis Method [12]-[16]

Consider a particular nonlinear differential equation that follows

$$\mathcal{N}[\mathfrak{H}(\Omega, t)] = 0 \quad (8)$$

Where $\mathcal{N}$ is a nonlinear operator, $\mathfrak{H}(\Omega, t)$ is an unknown function, and Ω might be either $\{\mathbf{x}, \eta\}$ or $\{\mathbf{x}, \eta, \mathfrak{z}\}$. Time and space are the two independent variables as represented by $\mathbf{x}, \eta, \mathfrak{z}$, and t respectively. Making use of the traditional Homotopy technique, which Liao devised.

$$(1-p)L[\varphi(\Omega, t; p) - \mathfrak{H}_0(\Omega, t)] = p\hbar \mathcal{N}[\varphi(\Omega, t; p)] \quad (9)$$

Where L acts as an auxiliary linear operator, $\varphi(\Omega, t; p)$ indicates an unknown function, $\mathfrak{H}_0(\Omega, t)$ is the starting guess for $\mathfrak{H}(\Omega, t)$, $\hbar$ serves as a nonzero auxiliary parameter, and $p \in [0, 1]$ represents the embedding parameter. It is applicable if $p = 0$ and $p = 1$.

$$\varphi(\Omega, t; 0) = \mathfrak{H}_0(\Omega, t),$$

and

$$\varphi(\Omega, t; 1) = \mathfrak{H}(\Omega, t),$$

Thus, solution $\varphi(\Omega, t; p)$ deviates from the original guess $\mathfrak{S}_0(\Omega, t)$ to solution $\mathfrak{S}(\Omega, t)$ as p approaches from 0 to 1. When we extend the Taylor series $\varphi(\Omega, t; p)$ concerning p , we obtain

$$\varphi(\Omega, t; p) = \mathfrak{S}_0(\Omega, t) + \sum_{m=1}^{\infty} \mathfrak{S}_m(\Omega, t) p^m \quad (10)$$

Where,

$$w_m(\Omega, t) = \frac{1}{m!} \left. \frac{\partial^m \varphi(\Omega, t; p)}{\partial p^m} \right|_{p=0}$$

The sequence (10) convergence occurs at $p = 1$ whenever the auxiliary linear operator, auxiliary parameter $\hbar$, initial guess, and auxiliary function have all been chosen correctly.

$$\mathfrak{S}(\Omega, t) = \mathfrak{S}_0(\Omega, t) + \sum_{m=1}^{\infty} \mathfrak{S}_m(\Omega, t), \quad (11)$$

The original nonlinear equation ought to have a legitimate solution in this instance. According to expression (11), the governing equation might be obtained from the 0-order deformation equation (9) as well. Describe the function of the vector.

$$\overrightarrow{\mathfrak{S}_n} = \{\mathfrak{S}_0(\Omega, t), \mathfrak{S}_1(\Omega, t), \mathfrak{S}_2(\Omega, t) \dots \dots \mathfrak{S}_n(\Omega, t)\}$$

The m -times concerning the embedding parameter p , differentiating the zero-ordered deformation equation (9). Once $p = 0$ has been entered and divided by $m!$ the m th-order deformation equation is as follows:

$$\begin{aligned} & L[\mathfrak{S}_m(\Omega, t) - \chi_m \mathfrak{S}_{m-1}(\Omega, t)] \\ &= \hbar \mathcal{R}_m[\mathfrak{S}_{m-1}(\Omega, t)] \\ & [\mathfrak{S}_m(\Omega, t) - \chi_m \mathfrak{S}_{m-1}(\Omega, t)] \\ &= L^{-1}\{\hbar \mathcal{R}_m[\mathfrak{S}_{m-1}(\Omega, t)]\} \end{aligned} \quad (12)$$

The implies

$$\begin{aligned} & \mathfrak{S}_m(\Omega, t) \\ &= \chi_m \mathfrak{S}_{m-1}(\Omega, t) \\ &+ L^{-1}\{\hbar \mathcal{R}_m[\mathfrak{S}_{m-1}(\Omega, t)]\} \end{aligned} \quad (13)$$

Where

$$\begin{aligned} & \overrightarrow{\mathcal{R}_m(\mathfrak{S}_{m-1})} \\ &= \frac{1}{m-1!} \left. \frac{\partial^{m-1} \mathcal{N}[\varphi(\Omega, t; p)]}{\partial p^{m-1}} \right|_{p=0} \end{aligned} \quad (14)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (15)$$

4 Test Examples:

In this section, we shall employ the Rangaig transform and the standard Homotopy analysis method (HAM) to obtain the semi analytical solutions of the (2+1)-D and (3+1)-D models of the nonlinear PDE.

Example 1: Examine the following set of partial differential equations that are nonlinear.

$$\begin{cases} u_t - \frac{1}{2} v(u_x^2 + u_y^2) = 0, \\ v_t - \frac{1}{2} u(v_x^2 + v_y^2) = 0, \end{cases} \quad (16)$$

Depending on the initial condition.

$$u_0(x, y, 0) = e^{x+y} \quad (17)$$

$$v_0(x, y, 0) = e^{-x-y} \quad (18)$$

Assuming Exact Function

$$u(x, y, t) = e^{x+y-t}, \quad (19)$$

$$v(x, y, t) = e^{-x-y-t}, \quad (20)$$

Substituting equation (16) into the Rangaig transform both sides of the equation, we get.

$$\begin{cases} \Re[u_t] = \Re\left[\frac{1}{2} v(u_x^2 + u_y^2)\right] \\ \Re[v_t] = \Re\left[\frac{1}{2} u(v_x^2 + v_y^2)\right] \end{cases}$$

$$\begin{cases} (-1)\omega \Re[u] + \frac{1}{\omega} u(\mathfrak{x}, \mathfrak{y}, 0) = \Re\left[\frac{1}{2} \mathfrak{v}(u_{\mathfrak{x}}^2 + u_{\mathfrak{y}}^2)\right] \\ (-1)\omega \Re[\mathfrak{v}] + \frac{1}{\omega} \mathfrak{v}(\mathfrak{x}, \mathfrak{y}, 0) = \Re\left[\frac{1}{2} u(\mathfrak{v}_{\mathfrak{x}}^2 + \mathfrak{v}_{\mathfrak{y}}^2)\right] \\ \omega \Re[u] = \frac{1}{\omega} u(\mathfrak{x}, \mathfrak{y}, 0) - \Re\left[\frac{1}{2} \mathfrak{v}(u_{\mathfrak{x}}^2 + u_{\mathfrak{y}}^2)\right] \\ \omega \Re[\mathfrak{v}] = \frac{1}{\omega} \mathfrak{v}(\mathfrak{x}, \mathfrak{y}, 0) - \Re\left[\frac{1}{2} u(\mathfrak{v}_{\mathfrak{x}}^2 + \mathfrak{v}_{\mathfrak{y}}^2)\right] \\ \Re[u] = \frac{1}{\omega^2} u(\mathfrak{x}, \mathfrak{y}, 0) - \frac{1}{\omega} \Re\left[\frac{1}{2} \mathfrak{v}(u_{\mathfrak{x}}^2 + u_{\mathfrak{y}}^2)\right] \\ \Re[\mathfrak{v}] = \frac{1}{\omega^2} \mathfrak{v}(\mathfrak{x}, \mathfrak{y}, 0) - \frac{1}{\omega} \Re\left[\frac{1}{2} u(\mathfrak{v}_{\mathfrak{x}}^2 + \mathfrak{v}_{\mathfrak{y}}^2)\right] \end{cases}$$

We get once basic conditions are applied

$$\begin{cases} \Re[u] = \frac{1}{\omega^2} e^{\mathfrak{x}+\mathfrak{y}} - \frac{1}{\omega} \Re\left[\frac{1}{2} \mathfrak{v}(u_{\mathfrak{x}}^2 + u_{\mathfrak{y}}^2)\right] \\ \Re[\mathfrak{v}] = \frac{1}{\omega^2} e^{-\mathfrak{x}-\mathfrak{y}} - \frac{1}{\omega} \Re\left[\frac{1}{2} u(\mathfrak{v}_{\mathfrak{x}}^2 + \mathfrak{v}_{\mathfrak{y}}^2)\right] \end{cases} \quad (21)$$

The nonlinear element is characterized as:

$$\begin{cases} N_u[\varphi_u(\mathfrak{x}, \mathfrak{y}, t; p)] = \Re[u] - \frac{1}{\omega^2} e^{\mathfrak{x}+\mathfrak{y}} - \frac{1}{\omega} \Re\left[\frac{1}{2} \mathfrak{v}(u_{\mathfrak{x}}^2 + u_{\mathfrak{y}}^2)\right] \\ N_v[\varphi_v(\mathfrak{x}, \mathfrak{y}, t; p)] = \Re[\mathfrak{v}] - \frac{1}{\omega^2} e^{-\mathfrak{x}-\mathfrak{y}} - \frac{1}{\omega} \Re\left[\frac{1}{2} u(\mathfrak{v}_{\mathfrak{x}}^2 + \mathfrak{v}_{\mathfrak{y}}^2)\right] \end{cases} \quad (22)$$

Here, we express the equation for zero-order deformation as follows:

$$\begin{cases} (1-p)\Re\{\varphi_u(\mathfrak{x}, \mathfrak{y}, t) - u_0(\mathfrak{x}, \mathfrak{y}, t)\} = pH_u(\mathfrak{x}, \mathfrak{y}, t)N_u[\varphi_u(\mathfrak{x}, \mathfrak{y}, t; p)] \\ (1-p)R\{\varphi_v(\mathfrak{x}, \mathfrak{y}, t) - \mathfrak{v}_0(\mathfrak{x}, \mathfrak{y}, t)\} = pH_v(\mathfrak{x}, \mathfrak{y}, t)N_v[\varphi_v(\mathfrak{x}, \mathfrak{y}, t; p)] \end{cases}$$

When $p = 0$ & $p = 1$, we have

$$\begin{cases} \varphi_u(\mathfrak{x}, \mathfrak{y}, t; 0) = u_0(\mathfrak{x}, \mathfrak{y}, 0) & \varphi_v(\mathfrak{x}, \mathfrak{y}, t; 0) = \mathfrak{v}_0(\mathfrak{x}, \mathfrak{y}, 0) \\ \varphi_u(\mathfrak{x}, \mathfrak{y}, t; 1) = u(\mathfrak{x}, \mathfrak{y}, t) & \varphi_v(\mathfrak{x}, \mathfrak{y}, t; 1) = \mathfrak{v}(\mathfrak{x}, \mathfrak{y}, t) \end{cases}$$

Thus, the equation for mth-order deformation

$$\begin{cases} \Re\{u_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m u_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = hH_u(\mathfrak{x}, \mathfrak{y}, t)\mathcal{R}_m(\overrightarrow{u_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \\ \Re\{\mathfrak{v}_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m \mathfrak{v}_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = hH_v(\mathfrak{x}, \mathfrak{y}, t)\mathcal{R}_m(\overrightarrow{\mathfrak{v}_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \end{cases} \quad (23)$$

The given assumption $H_u(\mathfrak{x}, \mathfrak{y}, t) = H_v(\mathfrak{x}, \mathfrak{y}, t) = 1$, in equ. (23). We have

$$\begin{cases} \Re\{u_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m u_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = h\mathcal{R}_m(\overrightarrow{u_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \\ \Re\{\mathfrak{v}_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m \mathfrak{v}_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = h\mathcal{R}_m(\overrightarrow{\mathfrak{v}_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \end{cases}$$

By inverse Rangaig transform both sides, we obtain

$$\begin{cases} u_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m u_{m-1}(\mathfrak{x}, \mathfrak{y}, t) = \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{u_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \\ \mathfrak{v}_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m \mathfrak{v}_{m-1}(\mathfrak{x}, \mathfrak{y}, t) = \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{\mathfrak{v}_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \end{cases}$$

The implies

$$\begin{cases} u_m(\mathfrak{x}, \mathfrak{y}, t) = \chi_m u_{m-1}(\mathfrak{x}, \mathfrak{y}, t) + \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{u_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \\ \mathfrak{v}_m(\mathfrak{x}, \mathfrak{y}, t) = \chi_m \mathfrak{v}_{m-1}(\mathfrak{x}, \mathfrak{y}, t) + \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{\mathfrak{v}_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \end{cases} \quad (24)$$

Find the value for $(u_0, u_1, u_2, u_3 \dots)$

From Equation (24) (Taking $h = -1, m = 0, 1, 2, 3 \dots$). We obtain

$$\begin{aligned} u_0(x, y, t) &= \overrightarrow{u_0}(x, y, t) \\ u_1(x, y, t) &= -\mathfrak{R}^{-1}\{\mathcal{R}_1(\overrightarrow{u_0}(x, y, t))\} \\ u_2(x, y, t) &= u_1(x, y, t) - \mathfrak{R}^{-1}\{\mathcal{R}_2(\overrightarrow{u_1}(x, y, t))\} \\ u_3(x, y, t) &= u_2(x, y, t) - \mathfrak{R}^{-1}\{\mathcal{R}_3(\overrightarrow{u_2}(x, y, t))\} \\ &\vdots \end{aligned}$$

Where

$$\mathcal{R}_m(\overrightarrow{u_{m-1}}) = \frac{1}{m-1!} \frac{\partial^{m-1} N_u[\varphi_u(x, y, t; p)]}{\partial p^{m-1}} \bigg|_{p=0} \quad (25)$$

Using equation (25), put $m = 1, 2, 3 \dots$, we obtain

$$\begin{aligned} \mathfrak{R}_1(\overrightarrow{u_0}(x, y, t)) &= \frac{1}{\omega^3} e^{x+y}, \\ \mathfrak{R}_2(\overrightarrow{u_1}(x, y, t)) &= \left(-\frac{1}{\omega^3} - \frac{1}{\omega^4}\right) e^{x+y}, \\ \mathfrak{R}_3(\overrightarrow{u_2}(x, y, t)) &= \left(-\frac{1}{\omega^4} - \frac{1}{\omega^5}\right) e^{x+y}, \\ &\vdots \end{aligned}$$

Therefore

$$\begin{aligned} u_0(x, y, t) &= e^{x+y}, \\ u_1(x, y, t) &= t \cdot e^{x+y}, \\ u_2(x, y, t) &= \frac{t^2}{2!} \cdot e^{x+y}, \\ u_3(x, y, t) &= \frac{t^3}{3!} e^{x+y}, \\ &\vdots \end{aligned}$$

The solution is:

$$u(x, y, t) = u_0 + u_1 + u_2 + \dots$$

Or

The final approximate solution for u .

$$u(x, y, t) = \left\{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right\} e^{x+y} = e^t \cdot e^{x+y} \quad (26)$$

$$u(x, y, t) = e^{x+y+t} \quad (27)$$

Figure 1 shows the physical behaviour of solutions u at $t = 0.5$ for different ranges of x and y , respectively.

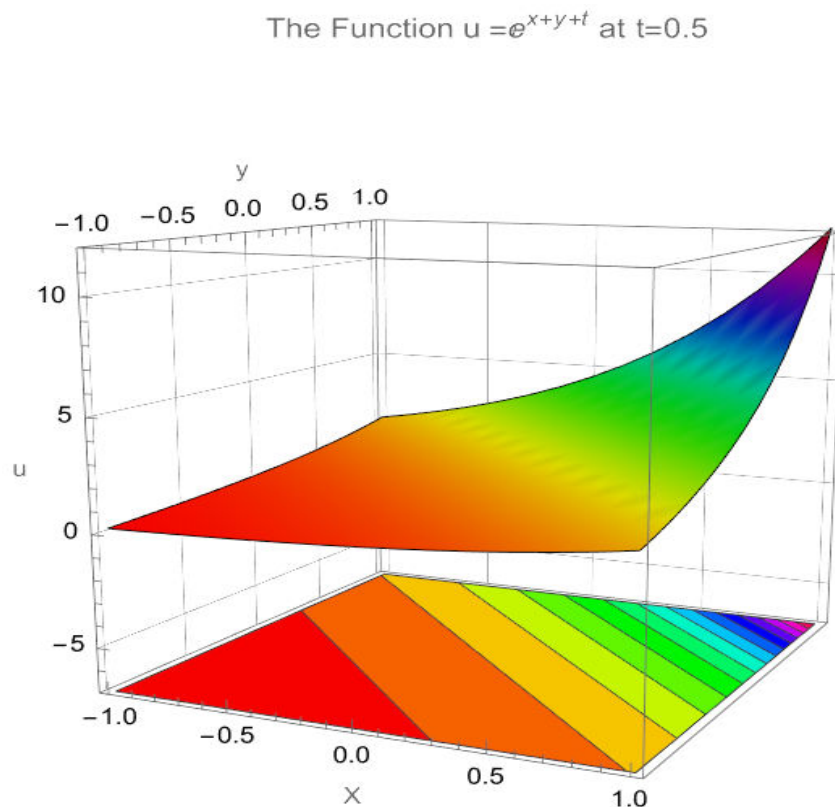


Figure 1: The physical behaviour of the solution “ u ” of Example 1 at $t = 0.5$ on surface.

Table 1: Comparison of “exact solution and approximate solution” form get the function $u(x, y, t)$ with the help of RT-HAM at different ranges of x and y at $t = 1$.

x	y	Exact Solution (u)	Approximate Solution (u^*)	Absolute Error (u)=
				$ u-u^* $
0.1	0.1	3.320114243	3.320112740	1.51×10^{-6}
0.2	0.2	4.055196148	4.055194313	1.83×10^{-6}
0.3	0.3	4.953027094	4.953024852	2.24×10^{-6}
0.4	0.4	6.049640140	6.049637402	2.74×10^{-6}
0.5	0.5	7.389046158	7.389042814	3.34×10^{-6}
0.6	0.6	9.025000144	9.024996060	4.08×10^{-6}
0.7	0.7	11.023158585	11.023153597	4.99×10^{-6}
0.8	0.8	13.463714488	13.463708395	6.09×10^{-6}

Table 1 Shows “the comparison of the exact and approximate solutions” of example 1 at different ranges of x and y at $t = 1$. The solutions obtained via “RT-HAM” are very close to the exact solutions.

Find the value for $(v_0, v_1, v_2, v_3 \dots)$

From Equation (24), (Taking $h = -1, m = 0, 1, 2, 3, \dots$). We obtain

$$\begin{aligned}v_0(\mathfrak{x}, \mathfrak{y}, t) &= \overrightarrow{v_0}(\mathfrak{x}, \mathfrak{y}, t), \\v_1(\mathfrak{x}, \mathfrak{y}, t) &= -\mathfrak{R}^{-1}\{\mathcal{R}_1(\overrightarrow{v_0}(\mathfrak{x}, \mathfrak{y}, t))\}, \\v_2(\mathfrak{x}, \mathfrak{y}, t) &= v_1(\mathfrak{x}, \mathfrak{y}, t) - \mathfrak{R}^{-1}\{\mathcal{R}_2(\overrightarrow{v_1}(\mathfrak{x}, \mathfrak{y}, t))\}, \\v_3(\mathfrak{x}, \mathfrak{y}, t) &= v_2(\mathfrak{x}, \mathfrak{y}, t) - \mathfrak{R}^{-1}\{\mathcal{R}_3(\overrightarrow{v_2}(\mathfrak{x}, \mathfrak{y}, t))\}, \\&\vdots\end{aligned}$$

Where

$$\mathcal{R}_m(\overrightarrow{u_{m-1}}) = \frac{1}{m-1!} \frac{\partial^{m-1} N_u[\varphi_u(\mathfrak{x}, \mathfrak{y}, t; p)]}{\partial p^{m-1}} \bigg|_{p=0} \quad (28)$$

Using equation (28), put $m = 1, 2, 3, \dots$, we obtain

$$\begin{aligned}\mathfrak{R}_1(\overrightarrow{v_0}(\mathfrak{x}, \mathfrak{y}, t)) &= \frac{1}{\omega^3} e^{-\mathfrak{x}-\mathfrak{y}}, \\\mathfrak{R}_2(\overrightarrow{v_1}(\mathfrak{x}, \mathfrak{y}, t)) &= \left(\frac{1}{\omega^3} - \frac{1}{\omega^4}\right) e^{-\mathfrak{x}-\mathfrak{y}}, \\\mathfrak{R}_3(\overrightarrow{v_2}(\mathfrak{x}, \mathfrak{y}, t)) &= \left(\frac{1}{\omega^4} - \frac{1}{\omega^5}\right) e^{-\mathfrak{x}-\mathfrak{y}}, \\&\vdots\end{aligned}$$

Therefore,

$$\begin{aligned}v_0(\mathfrak{x}, \mathfrak{y}, t) &= e^{-\mathfrak{x}-\mathfrak{y}}, \\v_1(\mathfrak{x}, \mathfrak{y}, t) &= -t \cdot e^{-\mathfrak{x}-\mathfrak{y}}, \\v_2(\mathfrak{x}, \mathfrak{y}, t) &= \frac{t^2}{2!} \cdot e^{-\mathfrak{x}-\mathfrak{y}}, \\v_3(\mathfrak{x}, \mathfrak{y}, t) &= -\frac{t^3}{3!} e^{-\mathfrak{x}-\mathfrak{y}}, \\&\vdots\end{aligned}$$

The solution is:

$$v(\mathfrak{x}, \mathfrak{y}, t) = v_0 + v_1 + v_2 + \dots$$

The final approximate solution for v .

$$v(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, t) = \left\{1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots\right\} e^{-\mathfrak{x}-\mathfrak{y}} = e^{-t} \cdot e^{-\mathfrak{x}-\mathfrak{y}} \quad (29)$$

$$v(\mathfrak{x}, \mathfrak{y}, t) = e^{-\mathfrak{x}-\mathfrak{y}-t} \quad (30)$$

Figure 2 shows the physical behaviour of solutions v at $t = 0.5$ for different ranges of $\mathfrak{x}$ and $\mathfrak{y}$, respectively.

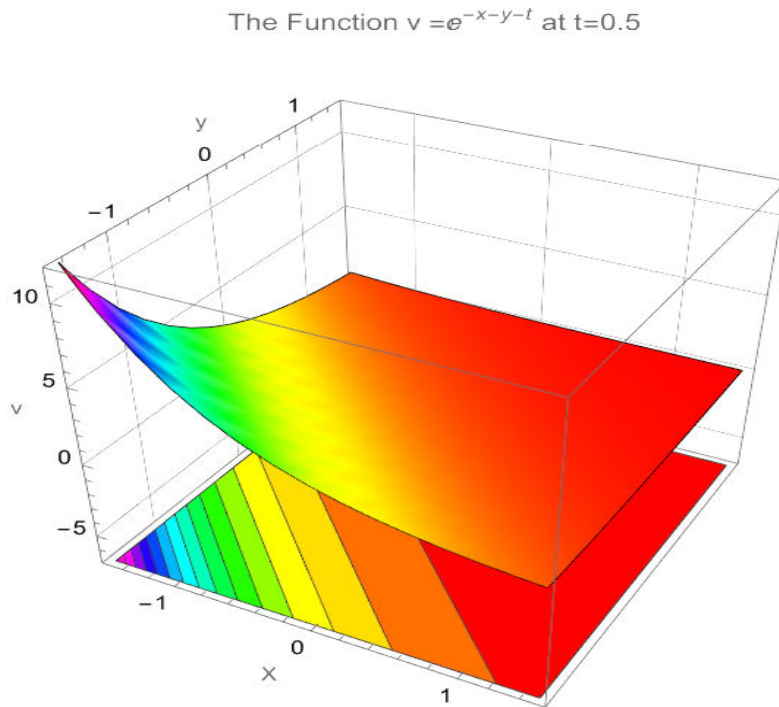


Figure 2: The physical behaviour of the solution “ v ” of Example 1 at $t = 0.5$ on surface.

Table 2: Comparison of “exact solution and approximate solution” form get the function (x, y, t) with the help of “RT-HAM” at different ranges of x and y at $t = 1$.

x	y	Exact Solution (v)	Approximate Solution (v^*)	Absolute Error ($v-v^*$)= $ v-v^* $
0.1	0.1	0.301194455	0.301196302	1.84×10^{-6}
0.2	0.2	0.246597196	0.246598708	1.51×10^{-6}
0.3	0.3	0.201896735	0.201897973	1.24×10^{-6}
0.4	0.4	0.165299088	0.165300102	1.01×10^{-6}
0.5	0.5	0.135335465	0.135336295	8.30×10^{-7}
0.6	0.6	0.110803322	0.110804002	6.79×10^{-7}
0.7	0.7	0.090718100	0.090718656	5.56×10^{-7}
0.8	0.8	0.074273708	0.074274164	4.55×10^{-7}

Table 2 Shows “the comparison of the exact and approximate solutions” of example 1 at different ranges of x and y at $t = 1$. The solutions obtained via “RT-HAM” are very close to the exact solutions.

Example 2: Consider the following system of nonlinear partial differential equations,

$$\begin{cases} u_t - v_x w_y = 1, \\ v_t - w_x u_y = 5, \\ w_t - u_x v_y = 5 \end{cases} \quad (31)$$

Depending on the initial condition.

$$u_0(x, y, 0) = x + 2y \quad (32)$$

$$v_0(x, y, 0) = x - 2y \quad (33)$$

$$w_0(x, y, 0) = -x + 2y \quad (34)$$

Assuming Exact Function

$$u(x, y, t) = x + 2y + 3t, \quad (35)$$

$$v(x, y, t) = x - 2y + 3t, \quad (36)$$

$$w(x, y, t) = -x + 2y + 3t, \quad (37)$$

Substituting equation (31), into the Rangaig transform both sides of the equation,

$$\begin{cases} \Re[u_t] = \Re[1 + v_x w_y] \\ \Re[v_t] = \Re[5 + w_x u_y] \\ \Re[w_t] = \Re[5 + u_x v_y] \end{cases}$$

The implies

$$\begin{cases} (-1)\omega \Re[u] + \frac{1}{\omega} u(x, y, 0) = \Re[1 + v_x w_y] \\ (-1)\omega \Re[v] + \frac{1}{\omega} v(x, y, 0) = \Re[5 + w_x u_y] \\ (-1)\omega \Re[w] + \frac{1}{\omega} w(x, y, 0) = \Re[5 + u_x v_y] \end{cases}$$

$$\begin{cases} \omega \Re[u] = \frac{1}{\omega} u(x, y, 0) - \Re[1 + v_x w_y] \\ \omega \Re[v] = \frac{1}{\omega} v(x, y, 0) - \Re[5 + w_x u_y] \\ \omega \Re[w] = \frac{1}{\omega} w(x, y, 0) - \Re[5 + u_x v_y] \end{cases}$$

$$\begin{cases} \Re[u] = \frac{1}{\omega^2} u(x, y, 0) - \frac{1}{\omega} \Re[1 + v_x w_y] \\ \Re[v] = \frac{1}{\omega^2} v(x, y, 0) - \frac{1}{\omega} \Re[5 + w_x u_y] \\ \Re[w] = \frac{1}{\omega^2} w(x, y, 0) - \frac{1}{\omega} \Re[5 + u_x v_y] \end{cases}$$

We get once basic conditions are applied

$$\begin{cases} \Re[u] = \frac{1}{\omega^2} (x + 2y) - \frac{1}{\omega} \Re[1 + v_x w_y] \\ \Re[v] = \frac{1}{\omega^2} (x - 2y) - \frac{1}{\omega} \Re[5 + w_x u_y] \\ \Re[w] = \frac{1}{\omega^2} (-x + 2y) - \frac{1}{\omega} \Re[5 + u_x v_y] \end{cases} \quad (38)$$

The nonlinear element is characterized as:

$$\begin{cases} N_u[\varphi_u(\mathfrak{x}, \mathfrak{y}, t; p)] = \Re[u] - \frac{1}{\omega^2}(\mathfrak{x} + 2\mathfrak{y}) - \frac{1}{\omega}\Re[1 + \mathfrak{v}_{\mathfrak{x}}w_{\mathfrak{y}}] \\ N_v[\varphi_v(\mathfrak{x}, \mathfrak{y}, t; p)] = \Re[v] - \frac{1}{\omega^2}(\mathfrak{x} - 2\mathfrak{y}) - \frac{1}{\omega}\Re[5 + w_{\mathfrak{x}}u_{\mathfrak{y}}] \\ N_w[\varphi_w(\mathfrak{x}, \mathfrak{y}, t; p)] = \Re[w] - \frac{1}{\omega^2}(-\mathfrak{x} + 2\mathfrak{y}) - \frac{1}{\omega}\Re[5 + u_{\mathfrak{x}}v_{\mathfrak{y}}] \end{cases} \quad (39)$$

Here, we express the equation for zero-order deformation as follows:

$$\begin{cases} (1-p)\Re\{\varphi_u(\mathfrak{x}, \mathfrak{y}, t) - u_0(\mathfrak{x}, \mathfrak{y}, t)\} = pH_u(\mathfrak{x}, \mathfrak{y}, t)N_u[\varphi_u(\mathfrak{x}, \mathfrak{y}, t; p)] \\ (1-p)\Re\{\varphi_v(\mathfrak{x}, \mathfrak{y}, t) - v_0(\mathfrak{x}, \mathfrak{y}, t)\} = pH_v(\mathfrak{x}, \mathfrak{y}, t)N_v[\varphi_v(\mathfrak{x}, \mathfrak{y}, t; p)] \\ (1-p)\Re\{\varphi_w(\mathfrak{x}, \mathfrak{y}, t) - w_0(\mathfrak{x}, \mathfrak{y}, t)\} = pH_w(\mathfrak{x}, \mathfrak{y}, t)N_w[\varphi_w(\mathfrak{x}, \mathfrak{y}, t; p)] \end{cases}$$

When $p = 0$ & $p = 1$, we have

$$\begin{cases} \varphi_u(\mathfrak{x}, \mathfrak{y}, t; 0) = u_0(\mathfrak{x}, \mathfrak{y}, 0) \\ \varphi_u(\mathfrak{x}, \mathfrak{y}, t; 1) = u(\mathfrak{x}, \mathfrak{y}, t) \end{cases} \quad \begin{cases} \varphi_v(\mathfrak{x}, \mathfrak{y}, t; 0) = v_0(\mathfrak{x}, \mathfrak{y}, 0) \\ \varphi_v(\mathfrak{x}, \mathfrak{y}, t; 1) = v(\mathfrak{x}, \mathfrak{y}, t) \end{cases} \quad \begin{cases} \varphi_w(\mathfrak{x}, \mathfrak{y}, t; 0) = w_0(\mathfrak{x}, \mathfrak{y}, 0) \\ \varphi_w(\mathfrak{x}, \mathfrak{y}, t; 1) = w(\mathfrak{x}, \mathfrak{y}, t) \end{cases}$$

Thus, the equation for m th-order deformation

$$\begin{cases} \Re\{u_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m u_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = hH_u(\mathfrak{x}, \mathfrak{y}, t)\mathcal{R}_m(\overrightarrow{u_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \\ \Re\{v_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m v_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = hH_v(\mathfrak{x}, \mathfrak{y}, t)\mathcal{R}_m(\overrightarrow{v_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \\ \Re\{w_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m w_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = hH_w(\mathfrak{x}, \mathfrak{y}, t)\mathcal{R}_m(\overrightarrow{w_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \end{cases} \quad (40)$$

The given assumption $H_u(\mathfrak{x}, \mathfrak{y}, t) = H_v(\mathfrak{x}, \mathfrak{y}, t) = H_w(\mathfrak{x}, \mathfrak{y}, t) = 1$, in equ.(40). We have

$$\begin{cases} \Re\{u_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m u_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = h\mathcal{R}_m(\overrightarrow{u_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \\ \Re\{v_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m v_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = h\mathcal{R}_m(\overrightarrow{v_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \\ \Re\{w_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m w_{m-1}(\mathfrak{x}, \mathfrak{y}, t)\} = h\mathcal{R}_m(\overrightarrow{w_{m-1}}(\mathfrak{x}, \mathfrak{y}, t)) \end{cases}$$

We get what we need by inverting the Rangaig transform on both sides.

$$\begin{cases} u_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m u_{m-1}(\mathfrak{x}, \mathfrak{y}, t) = \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{u_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \\ v_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m v_{m-1}(\mathfrak{x}, \mathfrak{y}, t) = \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{v_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \\ w_m(\mathfrak{x}, \mathfrak{y}, t) - \chi_m w_{m-1}(\mathfrak{x}, \mathfrak{y}, t) = \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{w_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \end{cases}$$

This implies

$$\begin{cases} u_m(\mathfrak{x}, \mathfrak{y}, t) = \chi_m u_{m-1}(\mathfrak{x}, \mathfrak{y}, t) + \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{u_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \\ v_m(\mathfrak{x}, \mathfrak{y}, t) = \chi_m v_{m-1}(\mathfrak{x}, \mathfrak{y}, t) + \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{v_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \\ w_m(\mathfrak{x}, \mathfrak{y}, t) = \chi_m w_{m-1}(\mathfrak{x}, \mathfrak{y}, t) + \Re^{-1}\{h\mathcal{R}_m(\overrightarrow{w_{m-1}}(\mathfrak{x}, \mathfrak{y}, t))\} \end{cases} \quad (41)$$

Find the value for $(u_0, u_1, u_2, u_3 \dots)$ from equation (41), (Taking, $m = 0, 1, 2, 3 \dots$).

$$\begin{aligned} u_0(\mathfrak{x}, \mathfrak{y}, t) &= \overrightarrow{u_0}(\mathfrak{x}, \mathfrak{y}, t), \\ u_1(\mathfrak{x}, \mathfrak{y}, t) &= -\Re^{-1}\{\mathcal{R}_1(\overrightarrow{u_0}(\mathfrak{x}, \mathfrak{y}, t))\} \\ u_2(\mathfrak{x}, \mathfrak{y}, t) &= u_1(\mathfrak{x}, \mathfrak{y}, t) - \Re^{-1}\{\mathcal{R}_2(\overrightarrow{u_1}(\mathfrak{x}, \mathfrak{y}, t))\} \\ u_3(\mathfrak{x}, \mathfrak{y}, t) &= u_2(\mathfrak{x}, \mathfrak{y}, t) - \Re^{-1}\{\mathcal{R}_3(\overrightarrow{u_2}(\mathfrak{x}, \mathfrak{y}, t))\} \\ &\vdots \end{aligned}$$

Where

$$\mathcal{R}_m(\overrightarrow{u_{m-1}}) = \frac{1}{m-1!} \frac{\partial^{m-1} N_u[\varphi_u(\mathfrak{x}, \mathfrak{y}, t; p)]}{\partial p^{m-1}} \Big|_{p=0} \quad (42)$$

Using equation (42), (Put $m = 1, 2, 3 \dots$), we obtain

$$\begin{aligned} \mathfrak{R}_1(\overrightarrow{u_0}(\mathfrak{x}, \mathfrak{y}, t)) &= \frac{3}{\omega^3}, \\ \mathfrak{R}_2(\overrightarrow{u_1}(\mathfrak{x}, \mathfrak{y}, t)) &= \frac{3h}{\omega^3}, \\ \mathfrak{R}_3(\overrightarrow{u_2}(\mathfrak{x}, \mathfrak{y}, t)) &= \frac{3h(1+h)}{\omega^3}, \\ &\vdots \end{aligned}$$

Therefore

$$\begin{aligned} u_0(\mathfrak{x}, \mathfrak{y}, t) &= \mathfrak{x} + 2\mathfrak{y}, \\ u_1(\mathfrak{x}, \mathfrak{y}, t) &= -3h.t, \\ u_2(\mathfrak{x}, \mathfrak{y}, t) &= -3h(1+h).t, \\ u_3(\mathfrak{x}, \mathfrak{y}, t) &= -3h(1+h)^2.t, \\ &\vdots \end{aligned}$$

The solution is:

$$u(\mathfrak{x}, \mathfrak{y}, t) = u_0 + u_1 + u_2 + \dots$$

Or

The final approximate solution for u .

$$u(\mathfrak{x}, \mathfrak{y}, t) = \{\mathfrak{x} + 2\mathfrak{y} + (-3h.t) + (-3h(1+h).t) + (-3h(1+h)^2.t) + \dots\} \quad (43)$$

From equation (43), (Taking, $h = -1$). Find the value of the exact solution.

$$u(\mathfrak{x}, \mathfrak{y}, t) = \mathfrak{x} + 2\mathfrak{y} + 3t \quad (44)$$

Figure 3 shows the physical behaviour of solutions u at $t = 0.5$ for different ranges of $\mathfrak{x}$ and $\mathfrak{y}$, respectively.

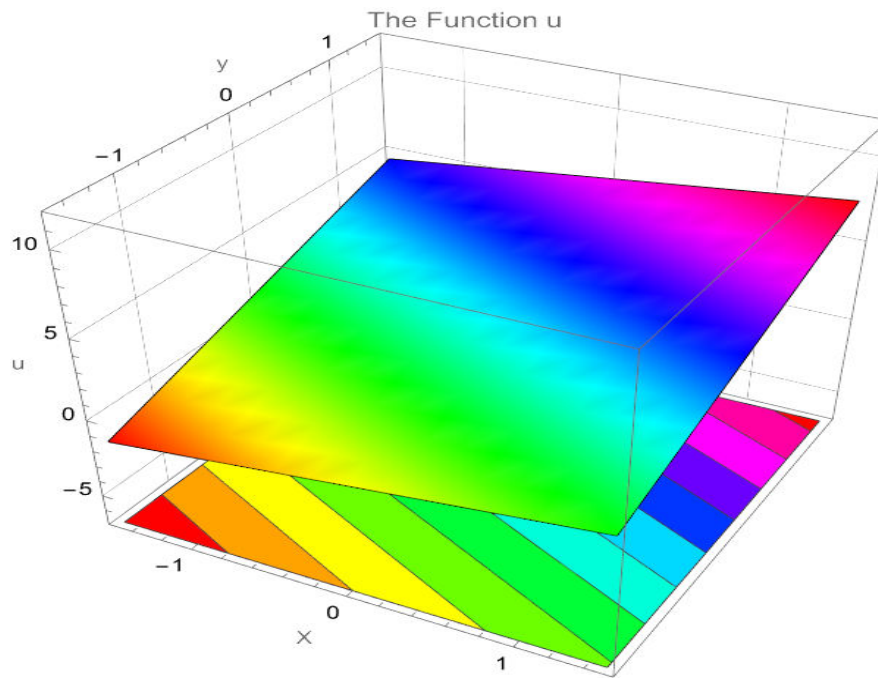


Figure 3: The physical behaviour of the solution “ u ” of Example 2 at $t = 1$ on surface.

Table 3: Comparison of “exact solution and approximate solution” form get the function $u(x, y, t)$ with the help of RT-HAM at different ranges of x and y at $t = 1$.

x	y	Exact Solution (u)	Approximate Solution (u^*)	Absolute Error (u)= $ u - u^* $
0.1	0.1	3.300000000	3.300000000	0
0.2	0.2	3.600000000	3.600000000	0
0.3	0.3	3.900000000	3.900000000	0
0.4	0.4	4.200000000	4.200000000	0
0.5	0.5	4.500000000	4.500000000	0
0.6	0.6	4.800000000	4.800000000	0
0.7	0.7	5.100000000	5.100000000	0
0.8	0.8	5.400000000	5.400000000	0

Table 3 Shows “the comparison of the exact and approximate solutions” of example 2 at different ranges of x and y at $t = 1$. The solutions obtained via “RT-HAM” are very close to the exact solutions.

Find the value of $(v_0, v_1, v_2, v_3 \dots)$ from equation (41), (Taking $m = 0, 1, 2, 3 \dots$).

$$v_0(x, y, t) = \overrightarrow{v_0}(x, y, t),$$

$$v_1(x, y, t) = -\mathcal{R}^{-1}\{\mathcal{R}_1(\overrightarrow{v_0}(x, y, t))\},$$

$$v_2(x, y, t) = v_1(x, y, t) - \mathcal{R}^{-1}\{\mathcal{R}_2(\overrightarrow{v_1}(x, y, t))\},$$

$$v_3(x, y, t) = v_2(x, y, t) - \mathcal{R}^{-1}\{\mathcal{R}_3(\overrightarrow{v_2}(x, y, t))\},$$

⋮

Where

$$\mathcal{R}_m(\overrightarrow{u_{m-1}}) = \frac{1}{m-1!} \frac{\partial^{m-1} N_u[\varphi_u(\mathfrak{x}, \mathfrak{y}, t; p)]}{\partial p^{m-1}} \Big|_{p=0} \quad (45)$$

Using equation (28), (Put $m = 1, 2, 3 \dots$), we obtain

$$\begin{aligned} \mathfrak{R}_1(\overrightarrow{v_0}(\mathfrak{x}, \mathfrak{y}, t)) &= \frac{3}{\omega^3}, \\ \mathfrak{R}_2(\overrightarrow{v_1}(\mathfrak{x}, \mathfrak{y}, t)) &= \frac{3h}{\omega^3}, \\ \mathfrak{R}_3(\overrightarrow{v_2}(\mathfrak{x}, \mathfrak{y}, t)) &= \frac{3h(1+h)}{\omega^3}, \\ &\vdots \end{aligned}$$

Therefore,

$$\begin{aligned} v_0(\mathfrak{x}, \mathfrak{y}, t) &= \mathfrak{x} - 2\mathfrak{y}, \\ v_1(\mathfrak{x}, \mathfrak{y}, t) &= -h \cdot 3t, \\ v_2(\mathfrak{x}, \mathfrak{y}, t) &= -3h(1+h) \cdot t, \\ v_3(\mathfrak{x}, \mathfrak{y}, t) &= -3h(1+h)^2 \cdot t, \\ &\vdots \end{aligned}$$

The solution is:

$$v(\mathfrak{x}, \mathfrak{y}, t) = v_0 + v_1 + v_2 + \dots$$

The final approximate solution for v .

$$v(\mathfrak{x}, \mathfrak{y}, t) = \{\mathfrak{x} - 2\mathfrak{y} + (-3h \cdot t) + (-3h(1+h) \cdot t) + (-3h(1+h)^2 \cdot t) + \dots\} \quad (46)$$

From equation (46), (Taking, $h = -1$). Find the value of the exact solution.

$$v(\mathfrak{x}, \mathfrak{y}, t) = \mathfrak{x} - 2\mathfrak{y} + 3t \quad (47)$$

Figure 4 shows the physical behaviour of solutions v at $t = 0.5$ for different ranges of $\mathfrak{x}$ and $\mathfrak{y}$, respectively.

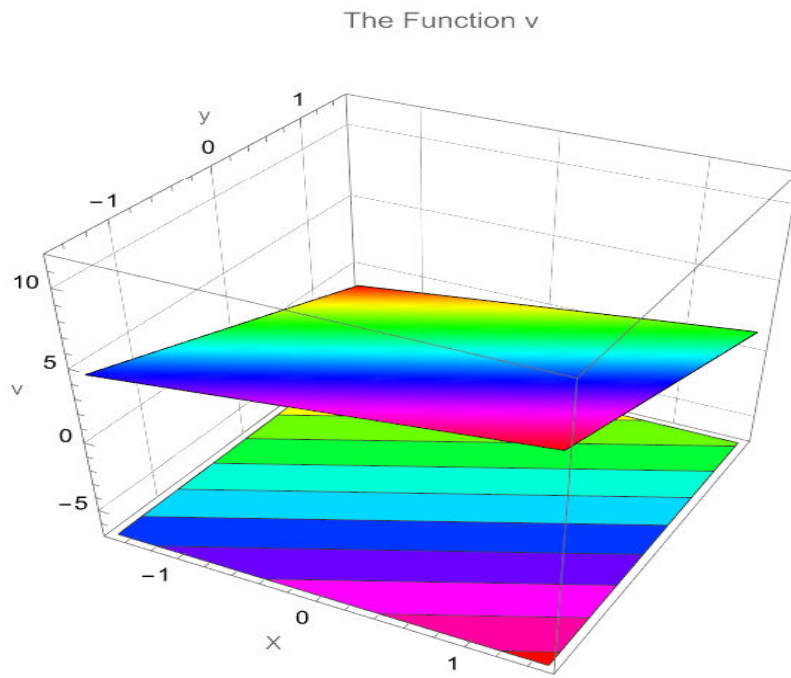


Figure 4: The physical behaviour of the solution “v” of Example 2 at $t = 1$ on surface.

Table 4: Comparison of “exact solution and approximate solution” form get the function (x, y, t) with the help of “RT-HAM” at different ranges of x and y at $t = 1$.

x	y	Exact Solution (v)	Approximate Solution (v^*)	Absolute Error ($v-v^*$)= $ v-v^* $
0.1	0.1	2.900000000	2.900000000	0
0.2	0.2	2.800000000	2.800000000	0
0.3	0.3	2.700000000	2.700000000	0
0.4	0.4	2.600000000	2.600000000	0
0.5	0.5	2.500000000	2.500000000	0
0.6	0.6	2.400000000	2.400000000	0
0.7	0.7	2.300000000	2.300000000	0
0.8	0.8	2.200000000	2.200000000	0

Table 4 Shows “the comparison of the exact and approximate solutions” of example 2 at different ranges of x and y at $t = 1$. The solutions obtained via “RT-HAM” are very close to the exact solutions.

Find the value of $(w_0, w_1, w_2, w_3 \dots)$ from equation (41), (Taking $m = 0, 1, 2, 3 \dots$).

$$w_0(x, y, t) = \overrightarrow{w_0}(x, y, t)$$

$$w_1(x, y, t) = -\mathfrak{R}^{-1}\{\mathcal{R}_1(\overrightarrow{w_0}(x, y, t))\}$$

$$w_2(x, y, t) = w_1(x, y, t) - \mathfrak{R}^{-1}\{\mathcal{R}_2(\overrightarrow{w_1}(x, y, t))\}$$

$$w_3(x, y, t) = w_2(x, y, t) - \mathfrak{R}^{-1}\{\mathcal{R}_3(\overrightarrow{w_2}(x, y, t))\}$$

⋮

Where

$$\mathcal{R}_m(\overrightarrow{w_{m-1}}) = \frac{1}{m-1!} \frac{\partial^{m-1} N_w[\varphi_w(\mathfrak{x}, \mathfrak{y}, t; p)]}{\partial p^{m-1}} \bigg|_{p=0} \quad (48)$$

Using equation (25), Put $m = 1, 2, 3 \dots$, we obtain

$$\mathfrak{R}_1(\overrightarrow{w_0}(\mathfrak{x}, \mathfrak{y}, t)) = \frac{3}{\omega^3},$$

$$\mathfrak{R}_2(\overrightarrow{w_1}(\mathfrak{x}, \mathfrak{y}, t)) = \frac{3h}{\omega^3},$$

$$\mathfrak{R}_3(\overrightarrow{w_2}(\mathfrak{x}, \mathfrak{y}, t)) = \frac{3h(1+h)}{\omega^3},$$

⋮

Therefore

$$w_0(\mathfrak{x}, \mathfrak{y}, t) = -\mathfrak{x} + 2\mathfrak{y},$$

$$w_1(\mathfrak{x}, \mathfrak{y}, t) = -3h.t,$$

$$w_2(\mathfrak{x}, \mathfrak{y}, t) = -3h(1+h).t,$$

$$w_3(\mathfrak{x}, \mathfrak{y}, t) = -3h(1+h)^2.t,$$

⋮

The solution is:

$$w(\mathfrak{x}, \mathfrak{y}, t) = w_0 + w_1 + w_2 + \dots$$

Or

The final approximate solution for w .

$$w(\mathfrak{x}, \mathfrak{y}, t) = \{-\mathfrak{x} + 2\mathfrak{y} + (-3h.t) + (-3h(1+h).t) + (-3h(1+h)^2.t) + \dots\} \quad (49)$$

From equation(49) (Taking, $h = -1$). Find the value of the exact solution.

$$w(\mathfrak{x}, \mathfrak{y}, t) = -\mathfrak{x} + 2\mathfrak{y} + 3t \quad (50)$$

Figure 5 shows the physical behaviour of solutions w at $t = 0.5$ for different ranges of x and y , respectively.

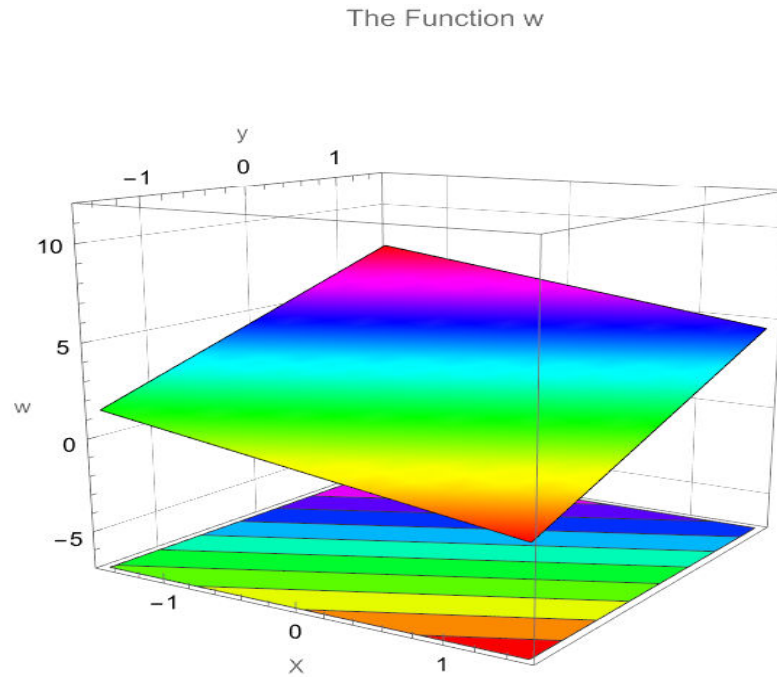


Figure 5: The physical behaviour of the solution “ w ” of Example 2 at $t = 1$ on surface.

Table 5: Comparison of “exact solution and approximate solution” form get the function $w(x, y, t)$ with the help of RT-HAM at different ranges of x and y at $t = 1$.

x	y	Exact Solution (w)	Approximate Solution (w^*)	Absolute Error (w)= $ w-w^* $
0.1	0.1	3.100000000	3.100000000	0
0.2	0.2	3.200000000	3.200000000	0
0.3	0.3	3.300000000	3.300000000	0
0.4	0.4	3.400000000	3.400000000	0
0.5	0.5	3.500000000	3.500000000	0
0.6	0.6	3.600000000	3.600000000	0
0.7	0.7	3.700000000	3.700000000	0
0.8	0.8	3.800000000	3.800000000	0

Table 5 Shows “the comparison of the exact and approximate solutions” of example 2 at different ranges of x and y at $t = 1$. The solutions obtained via “RT-HAM” are very close to the exact solutions.

Example 3: consider the following system of nonlinear partial differential equations,

$$\begin{cases} u_t + \frac{1}{3}v^2(u_x^2 + u_y^2 + u_z^2) + u = 1, \\ v_t + \frac{1}{3}u^2(v_x^2 + v_y^2 + v_z^2) - v = 1, \end{cases} \quad (51)$$

Depending on the initial condition.

$$u_0(x, y, z, 0) = e^{x+y+z} \quad (52)$$

$$v_0(x, y, z, 0) = e^{-x-y-z} \quad (53)$$

Assuming Exact Function

$$u(x, y, z, t) = e^{x+y+z-t}, \quad (54)$$

$$v(x, y, z, t) = e^{-x-y-z+t}, \quad (55)$$

Substituting equation (51), into the Rangaig transform both sides of the equation

$$\begin{cases} \mathfrak{R}[u_t] = \mathfrak{R}\left[1 - u - \frac{1}{3}v^2(u_x^2 + u_y^2 + u_z^2)\right] \\ \mathfrak{R}[v_t] = \mathfrak{R}\left[1 + v - \frac{1}{3}u^2(v_x^2 + v_y^2 + v_z^2)\right] \end{cases} \quad (56)$$

The implies

$$\begin{cases} (-1)\omega \mathfrak{R}[u] + \frac{1}{\omega} u(x, y, z, 0) = \mathfrak{R}\left[1 - u - \frac{1}{3}v^2(u_x^2 + u_y^2 + u_z^2)\right] \\ (-1)\omega \mathfrak{R}[v] + \frac{1}{\omega} v(x, y, z, 0) = \mathfrak{R}\left[1 + v - \frac{1}{3}u^2(v_x^2 + v_y^2 + v_z^2)\right] \\ \omega \mathfrak{R}[u] = \frac{1}{\omega} u(x, y, z, 0) - \mathfrak{R}\left[1 - u - \frac{1}{3}v^2(u_x^2 + u_y^2 + u_z^2)\right] \\ \omega \mathfrak{R}[v] = \frac{1}{\omega} v(x, y, z, 0) - \mathfrak{R}\left[1 + v - \frac{1}{3}u^2(v_x^2 + v_y^2 + v_z^2)\right] \\ \mathfrak{R}[u] = \frac{1}{\omega^2} u(x, y, z, 0) - \frac{1}{\omega} \mathfrak{R}\left[1 - u - \frac{1}{3}v^2(u_x^2 + u_y^2 + u_z^2)\right] \\ \mathfrak{R}[v] = \frac{1}{\omega^2} v(x, y, z, 0) - \frac{1}{\omega} \mathfrak{R}\left[1 + v - \frac{1}{3}u^2(v_x^2 + v_y^2 + v_z^2)\right] \end{cases}$$

We get once basic conditions are applied

$$\begin{cases} \mathfrak{R}[u] = \frac{1}{\omega^2} e^{x+y+z} - \frac{1}{\omega} \mathfrak{R}\left[1 - u - \frac{1}{3}v^2(u_x^2 + u_y^2 + u_z^2)\right] \\ \mathfrak{R}[v] = \frac{1}{\omega^2} e^{-x-y-z} - \frac{1}{\omega} \mathfrak{R}\left[1 + v - \frac{1}{3}u^2(v_x^2 + v_y^2 + v_z^2)\right] \end{cases} \quad (57)$$

The nonlinear element is characterized as:

$$\begin{cases} N_u[\varphi_u(x, y, z, t; p)] = \mathfrak{R}[u] - \frac{1}{\omega^2} e^{x+y+z} + \frac{1}{\omega} \mathfrak{R}\left[1 - u - \frac{1}{3}v^2(u_x^2 + u_y^2 + u_z^2)\right] \\ N_v[\varphi_v(x, y, z, t; p)] = \mathfrak{R}[v] - \frac{1}{\omega^2} e^{-x-y-z} + \frac{1}{\omega} \mathfrak{R}\left[1 + v - \frac{1}{3}u^2(v_x^2 + v_y^2 + v_z^2)\right] \end{cases} \quad (58)$$

Here, we express the equation for zero-order deformation as follows:

$$\begin{cases} (1-p)\mathfrak{R}\{\varphi_u(x, y, z, t) - u_0(x, y, z, t)\} = phH_u(x, y, z, t)N_u[\varphi_u(x, y, z, t; p)] \\ (1-p)\mathfrak{R}\{\varphi_v(x, y, z, t) - v_0(x, y, z, t)\} = phH_v(x, y, z, t)N_v[\varphi_v(x, y, z, t; p)] \end{cases}$$

When $p = 0$ & $p = 1$, we have

$$\begin{cases} \varphi_u(x, y, z, t; 0) = u_0(x, y, z, 0) & \varphi_v(x, y, z, t; 0) = v_0(x, y, z, 0) \\ \varphi_u(x, y, z, t; 1) = u(x, y, z, t) & \varphi_v(x, y, z, t; 1) = v(x, y, z, t) \end{cases}$$

Thus, the equation for mth-order deformation

$$\begin{cases} \Re\{u_m(x, y, z, t) - \chi_m u_{m-1}(x, y, z, t)\} = h H_u(x, y, z, t) \mathcal{R}_m(\overrightarrow{u_{m-1}}(x, y, z, t)) \\ \Re\{v_m(x, y, z, t) - \chi_m v_{m-1}(x, y, z, t)\} = h H_v(x, y, z, t) \mathcal{R}_m(\overrightarrow{v_{m-1}}(x, y, z, t)) \end{cases} \quad (59)$$

The given assumption $H_u(x, y, z, t) = H_v(x, y, z, t) = 1$, in equ.(59), we have

$$\begin{cases} \Re\{u_m(x, y, z, t) - \chi_m u_{m-1}(x, y, z, t)\} = h \mathcal{R}_m(\overrightarrow{u_{m-1}}(x, y, z, t)) \\ \Re\{v_m(x, y, z, t) - \chi_m v_{m-1}(x, y, z, t)\} = h \mathcal{R}_m(\overrightarrow{v_{m-1}}(x, y, z, t)) \end{cases}$$

We get what we need by inverting the Rangaig transform on both sides.

$$\begin{cases} u_m(x, y, z, t) - \chi_m u_{m-1}(x, y, z, t) = \Re^{-1}\{h \mathcal{R}_m(\overrightarrow{u_{m-1}}(x, y, z, t))\} \\ v_m(x, y, z, t) - \chi_m v_{m-1}(x, y, z, t) = \Re^{-1}\{h \mathcal{R}_m(\overrightarrow{v_{m-1}}(x, y, z, t))\} \end{cases}$$

The implies

$$\begin{cases} u_m(x, y, z, t) = \chi_m u_{m-1}(x, y, z, t) + \Re^{-1}\{h \mathcal{R}_m(\overrightarrow{u_{m-1}}(x, y, z, t))\} \\ v_m(x, y, z, t) = \chi_m v_{m-1}(x, y, z, t) + \Re^{-1}\{h \mathcal{R}_m(\overrightarrow{v_{m-1}}(x, y, z, t))\} \end{cases} \quad (60)$$

Find the value for $(u_0, u_1, u_2, u_3 \dots)$

From Equation (60), (Taking $h = -1, m = 0, 1, 2, 3 \dots$). We obtain

$$\begin{aligned} u_0(x, y, z, t) &= \overrightarrow{u_0}(x, y, z, t) \\ u_1(x, y, z, t) &= -\Re^{-1}\{\mathcal{R}_1(\overrightarrow{u_0}(x, y, z, t))\} \\ u_2(x, y, z, t) &= u_1(x, y, z, t) - \Re^{-1}\{\mathcal{R}_2(\overrightarrow{u_1}(x, y, z, t))\} \\ u_3(x, y, z, t) &= u_2(x, y, z, t) - \Re^{-1}\{\mathcal{R}_3(\overrightarrow{u_2}(x, y, z, t))\} \\ &\vdots \end{aligned}$$

Where

$$\mathcal{R}_m(\overrightarrow{u_{m-1}}) = \frac{1}{m-1!} \frac{\partial^{m-1} N_u[\varphi_u(x, y, z, t; p)]}{\partial p^{m-1}} \Big|_{p=0} \quad (61)$$

Using equation (61), put $m = 1, 2, 3 \dots$, we obtain

$$\begin{aligned} \Re_1(\overrightarrow{u_0}(x, y, z, t)) &= -\frac{1}{\omega^3} e^{x+y+z}, \\ \Re_2(\overrightarrow{u_1}(x, y, z, t)) &= \left(\frac{1}{\omega^3} - \frac{1}{\omega^4}\right) e^{x+y+z}, \\ \Re_3(\overrightarrow{u_2}(x, y, z, t)) &= \left(\frac{1}{\omega^4} - \frac{1}{\omega^5}\right) e^{x+y+z}, \\ &\vdots \end{aligned}$$

Therefore,

$$\begin{aligned} u_0(x, y, z, t) &= e^{x+y+z}, \\ u_1(x, y, z, t) &= -t \cdot e^{x+y+z}, \\ u_2(x, y, z, t) &= \frac{t^2}{2!} \cdot e^{x+y+z}, \\ u_3(x, y, z, t) &= -\frac{t^3}{3!} e^{x+y+z}, \\ &\vdots \end{aligned}$$

The solution is:

$$u(x, y, z, t) = u_0 + u_1 + u_2 + \dots$$

Or

The final approximate solution for u .

$$u(x, y, z, t) = \left\{ 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right\} e^{x+y+z} = e^{-t} \cdot e^{x+y+z} \quad (62)$$

$$u(x, y, z, t) = \quad (63)$$

Figure 6 shows the physical behaviour of solutions u at $t = 0.5$ for different ranges of x, y and z , respectively.

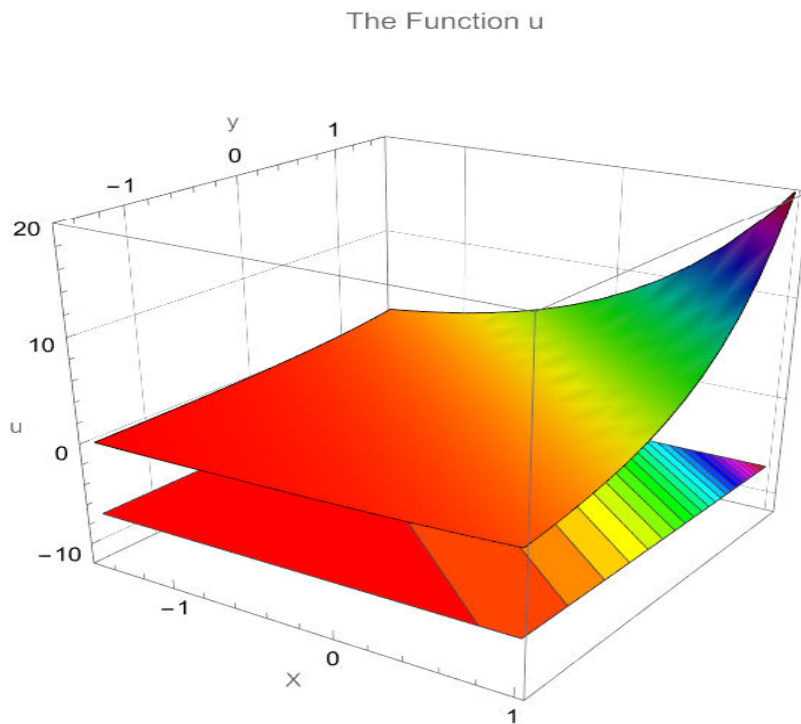


Figure 6: The physical behaviour of the solution “ u ” of Example 3 at $z = 1$ and $t = 0.5$ on surface

Table 6: Comparison of “exact solution and approximate solution” form get the function $u(x, y, z, t)$ with the help of “RT-HAM” at different ranges of x, y and z at $t = 1$.

x	y	z	Exact Solution (u)	Approximate Solution (u^*)	Absolute Error (u)= $ u-u^* $
0.1	0.1	0.1	2.013751759	2.013750848	9.11×10^{-7}
0.2	0.2	0.2	1.491824296	1.491823621	6.75×10^{-7}
0.3	0.3	0.3	1.105170844	1.105170344	5.00×10^{-7}
0.4	0.4	0.4	0.818730863	0.818730493	3.71×10^{-7}
0.5	0.5	0.5	0.606530864	0.606530589	2.74×10^{-7}
0.6	0.6	0.6	0.449329206	0.449329003	2.03×10^{-7}
0.7	0.7	0.7	0.332871330	0.332871179	1.51×10^{-7}
0.8	0.8	0.8	0.246597196	0.246597085	1.12×10^{-7}

Table 6 Shows “the comparison of the exact and approximate solutions” of example 3 at different ranges of x, y and z at $t = 1$. The solutions obtained via “RT-HAM” are very close to the exact solutions.

Find the value for $(v_0, v_1, v_2, v_3 \dots)$

From Equation (60), (Taking $h = -1, m = 0, 1, 2, 3 \dots$). We obtain

$$\begin{aligned}v_0(x, y, z, t) &= \overrightarrow{v_0}(x, y, z, t), \\v_1(x, y, z, t) &= -\Re^{-1}\{\mathcal{R}_1(\overrightarrow{v_0}(x, y, z, t))\}, \\v_2(x, y, z, t) &= v_1(x, y, z, t) - \Re^{-1}\{\mathcal{R}_2(\overrightarrow{v_1}(x, y, z, t))\}, \\v_3(x, y, z, t) &= v_2(x, y, z, t) - \Re^{-1}\{\mathcal{R}_3(\overrightarrow{v_2}(x, y, z, t))\}, \\&\vdots\end{aligned}$$

Where

$$\mathcal{R}_m(\overrightarrow{u_{m-1}}) = \frac{1}{m-1!} \frac{\partial^{m-1} N_u[\varphi_u(x, y, t; p)]}{\partial p^{m-1}} \bigg|_{p=0} \quad (64)$$

Using equation (64), put $m = 1, 2, 3 \dots$, we obtain

$$\begin{aligned}\Re_1(\overrightarrow{v_0}(x, y, z, t)) &= \frac{1}{\omega^3} e^{-x-y-z}, \\\Re_2(\overrightarrow{v_1}(x, y, z, t)) &= \left(\frac{1}{\omega^3} - \frac{1}{\omega^4}\right) e^{-x-y-z}, \\\Re_3(\overrightarrow{v_2}(x, y, z, t)) &= \left(\frac{1}{\omega^4} - \frac{1}{\omega^5}\right) e^{-x-y-z}, \\&\vdots\end{aligned}$$

Therefore

$$\begin{aligned}v_0(x, y, z, t) &= e^{-x-y-z}, \\v_1(x, y, z, t) &= t \cdot e^{-x-y-z}, \\v_2(x, y, z, t) &= \frac{t^2}{2!} \cdot e^{-x-y-z}, \\v_3(x, y, z, t) &= \frac{t^3}{3!} e^{-x-y-z}, \\&\vdots\end{aligned}$$

The solution is:

$$v(x, y, z, t) = v_0 + v_1 + v_2 + \dots$$

Or

The final approximate solution for v .

$$v(x, y, z, t) = \left\{ 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right\} e^{-x-y-z} = e^t \cdot e^{-x-y-z} \quad (65)$$

$$v(x, y, z, t) = e^{-x-y-z+t} \quad (66)$$

Figure 7 shows the physical behaviour of solutions x, y and z at $t = 0.5$, respectively.

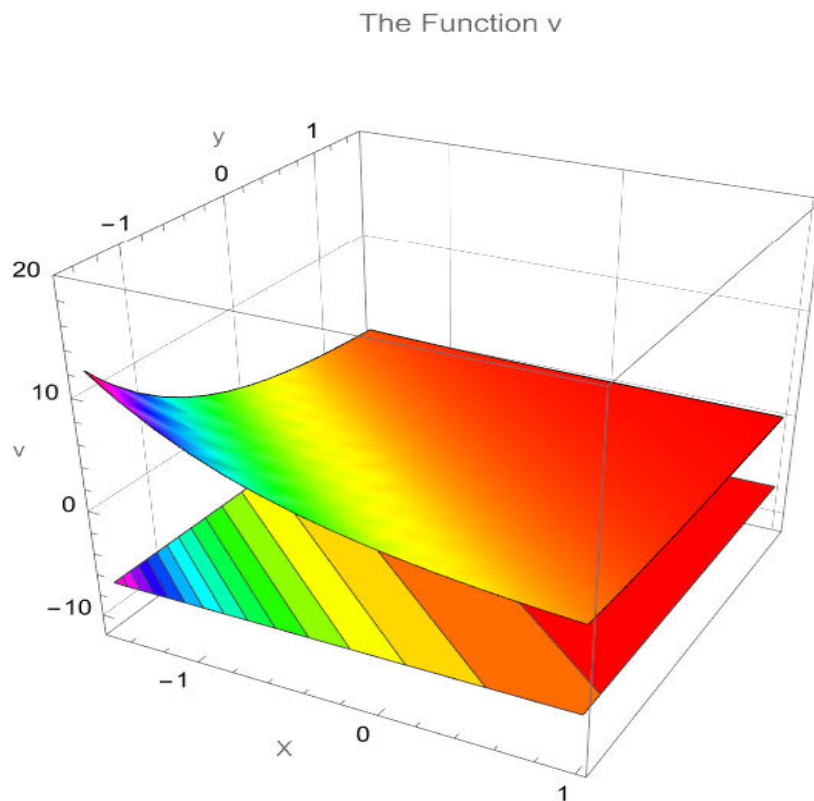


Figure 7: The physical behaviour of the solution “ v ” of Example 3 at $z = 1$ and $t = 0.5$ on surface

Table 7: Comparison of “exact solution and approximate solution” form get the function $v(x, y, z, t)$ with the help of “RT-HAM” at different ranges of x, y and z at $t = 1$.

x	y	z	Exact Solution (v)	Approximate Solution (v^*)	Absolute Error (v) = $ v-v^* $
0.1	0.1	0.1	0.496585538	0.496588583	3.04×10^{-6}
0.2	0.2	0.2	0.670320226	0.670324337	4.11×10^{-6}
0.3	0.3	0.3	0.904837479	0.904843027	5.55×10^{-6}
0.4	0.4	0.4	1.221402594	1.221410083	7.49×10^{-6}
0.5	0.5	0.5	1.648720716	1.648730826	1.01×10^{-5}
0.6	0.6	0.6	2.225539731	2.225553378	1.36×10^{-5}
0.7	0.7	0.7	3.004163801	3.004182222	1.84×10^{-5}
0.8	0.8	0.8	4.055196148	4.055221014	2.49×10^{-5}

Table 7 shows “the comparison of the exact and approximate solutions” of example 3 at different ranges of x, y and z at $t = 1$. The solutions obtained via “RT-HAM” are very close to the exact solutions.

5 Conclusion

All things considered, the above-mentioned visual and numerical evidence points to the suggested scheme as a straightforward and precise semi-analytical approach to solving systems of nonlinear (2+1) D and (3+1) D partial differential equations. Applications of this method to nonlinear models of partial differential equations include a wide range of practical issues.

Conflicts of interest

There is no conflict of interest regarding this paper.

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