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Rigorous Analytical and Numerical Analysis of Unsteady Free Convection Flow Over a Stretching Sheet with Thermal Radiation and Chemical Reaction.

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Abstract

This paper presents a comprehensive analytical and numerical study of unsteady free convection flow over a continuously stretching sheet under the simultaneous influences of thermal radiation and a first–order chemical reaction. The governing equations are derived under the Boussinesq approximation and reduced via similarity transformations 1 to a set of nonlinear ordinary differential equations. Advanced analytical techniques, including the Adomian Decomposition Method (ADM) and the Homotopy Analysis Method (HAM), are employed to obtain convergent series solutions. Rigorous convergence analysis and residual error estimates are provided, and the analytical results are validated by a high–accuracy Runge–Kutta–Fehlberg shooting method. Extensive parametric studies are conducted to examine the effects of magnetic damping, radiative heat transfer, and chemical reaction on the velocity, thermal, and concentration profiles. The findings offer deep insight into the complex interactions among multiple physical processes and serve as a benchmark for future investigations.

1. Introduction

Fluid flow over a stretching sheet is a classical problem that arises in many industrial applications such as polymer extrusion, glass fiber production, and cooling of metallic sheets. In practical systems, the flow is affected not only by the mechanical stretching of the sheet but also by thermal effects (free convection and thermal radiation) and chemical reactions. Despite the extensive literature on individual aspects of the problem (see, e.g., [1,2]), a unified study that incorporates all these effects remains sparse.

This paper presents a rigorous analytical treatment that integrates the effects of free convection, thermal radiation, viscous dissipation, and a first–order chemical reaction into a unified mathematical model. The governing partial differential equations (PDEs) are reduced via similarity transformations to

a nonlinear ordinary differential equation (ODE). We then apply advanced analytical methods—notably the Adomian Decomposition Method (ADM) and the Homotopy Analysis Method (HAM)—to derive convergent series solutions. The analytical findings are further validated through numerical simulations using a Runge–Kutta–Fehlberg shooting method. An extensive parametric study demonstrates how key parameters influence the flow characteristics.

2. Mathematical Formulation and Similarity Transformation

2.1 Governing Equations and Physical Assumptions

We consider the unsteady, two-dimensional flow of an incompressible, Newtonian fluid over a stretching sheet. The sheet's velocity is assumed to be proportional to the distance from the origin

$$U(x^*, t^*) = a(t^*)x^*$$

where $a(t^*)$ is a time-dependent stretching rate. Free convection is induced by buoyancy forces resulting from the temperature difference between the stretching sheet (temperature $T_w(t^*)$) and the ambient fluid (temperature T_∞). Additionally, thermal radiation, modeled using the Rosseland approximation, and a first–order chemical reaction affecting the species concentration are taken into account. Under the Boussinesq approximation and assuming constant fluid properties (except for density variations in buoyancy), the governing equations in dimensional form are:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{1}$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \nu \frac{\partial^2 u^*}{\partial y^{*2}} + g\beta_T (T^* - T_\infty) \tag{2}$$

$$\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \alpha \frac{\partial^2 T^*}{\partial y^{*2}} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y^{*2}} \tag{3}$$

$$\frac{\partial C^*}{\partial t^*} + u^* \frac{\partial C^*}{\partial x^*} + v^* \frac{\partial C^*}{\partial y^*} = D \frac{\partial^2 C^*}{\partial y^{*2}} - k_r (C^* - C_\infty) \tag{4}$$

where u^* and v^* are the velocity components along x^* and y^*, respectively; T^* is the temperature; C^* is the species concentration; ν is the kinematic viscosity; α is the thermal diffusivity; D is the mass diffusivity; and k_r is the chemical reaction rate constant.

The radiative heat flux q_r is modelled by the Rosseland approximation:

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^{*4}}{\partial y^*} \tag{5}$$

which is linearized for small temperature differences:

$$T^{*4} \approx 4T_\infty^3 T^* - 3T_\infty^4 \tag{6}$$

2.2Non-Dimensionalization

We introduce the following non-dimensional variables:

$$x = \frac{x^*}{L},\ \ y = \frac{y^*}{\delta},\ \ t = \frac{t^*}{t_0},\ \ u = \frac{u^*}{U_0},\ \ v = \frac{v^*}{U_0} \tag{7}$$

$$\theta = \frac{T^* - T_\infty}{T_w - T_\infty}, \ \varphi = \frac{C^* - C_\infty}{C - C_\infty}, \tag{8}$$

where L is a characteristic length, t_0 is a characteristic time, δ is the boundary layer thickness, and U_0 is a reference velocity (typically U_0 = aL for a stretching sheet).

2.3 Similarity Transformation

Following Crane [1], we define the similarity variable:

$$\eta = \sqrt{\frac{a}{\nu}}\, y^* \tag{9}$$

and introduce the stream function $\psi(x^*,y^*,t^*)$ such that

$$u^* = \frac{\partial \psi}{\partial y^*}, \quad v^* = -\frac{\partial \psi}{\partial x^*} \tag{10}$$

We express the stream function as

$$\psi(x^*, y^*, t^*) = \sqrt{a\nu}\, x^* f(\eta) \tag{11}$$

Then the velocity components become:

$$u^* = a x^* f^{'}(\eta)\sqrt{\frac{a}{\nu}} \tag{12}$$

$$v^* = -\sqrt{a\nu}\, f(\eta) \tag{13}$$

After substituting these expressions into the momentum equation and applying the boundary layer approximations, we obtain the following non-linear ODE:

$$f'''(\eta) + f(\eta)f''(\eta) - \left[f'(\eta)\right]^2 + \lambda\theta(\eta) = 0 \quad (14)$$

with boundary conditions:

$$f(0) = 0, \ \ f'(0) = 1, \ \ f'(\infty) = 1 \quad (15)$$

Here, λ is a non-dimensional parameter representing the buoyancy effect.

3. Analytical Solution via the Adomian De-composition Method

3.1 Operator Formulation

We rewrite Eq. (14) in the operator form as:

$$L[f(\eta)] + N[f(\eta)] = 0$$

where the linear operator L is defined as

$$L[f(\eta)] = f'''(\eta)$$

and the nonlinear operator N is defined by

$$N[f(\eta)] = f(\eta)f''(\eta) - \left[f'(\eta)\right]^2 + \lambda\theta(\eta)$$

3.2Series Expansion

We assume that the solution f(η) can be expressed as an infinite series:

$$f(\eta) = \sum_{n=0}^{\infty} f_n(\eta)$$

The nonlinear operator is similarly decomposed into Adomian polynomials:

$$N\left[\sum_{n=0}^{\infty} f_n(\eta)\right] = \sum_{n=0}^{\infty} A_n(\eta)$$

$$A_n(\eta) = \frac{1}{n!}\frac{d^n}{dp^n} N\left[\Sigma_{k=0}^{\infty} p^k f_k(\eta)\right]_{p=0}$$

3.3Integral Representation

Since L is a third–order derivative, its inverse is given by a triple integration. Using the boundary conditions f(0) = 0 and f′(0) = 1, the integral form of the solution is:

$$f(\eta) = \eta + \frac{\beta}{2}\eta^2 - \int_0^{\eta} \frac{(\eta - s)^2}{2} N[f(s)]ds$$

Where $\beta = f''(0)$ is an unknown constant determined by enforcing $f'(\infty) = 1$.

3.4Recursive Scheme

Substitute the series expansion for f(η) and the Adomian decomposition into the integral form to obtain:

$$\sum_{n=0}^{\infty} f_n(\eta) = \eta + \frac{\beta}{2}\eta^2 - \int_0^{\eta} \frac{(\eta - s)^2}{2} \Sigma_{n=0}^{\infty} A_n(s)ds$$

Equate terms of like order to derive:

$$f_0(\eta) = \eta \tag{16}$$

$$f_{n+1}(\eta) = \int_0^\eta \frac{(\eta-s)^2}{2} A_n(s)ds, \quad n \geq 0 \tag{17}$$

3.5Computation of the First Adomian Polynomial

Since $f_0(\eta) = \eta$, we have

$$f_0'(\eta) = 1, \quad f_0''(\eta) = 0$$

Thus, the zeroth–order Adomian polynomial is:

$$A_0(\eta) = f_0(\eta)f_0''(\eta) - \left[f_0'(\eta)\right]^2 + \lambda\theta(\eta) = -1 + \lambda\theta(\eta)$$

Therefore, the first–order correction is:

$$f_1(\eta) = -\int_0^\eta \frac{(\eta-s)^2}{2} A_0(s)ds = \int_0^\eta \frac{(\eta-s)^2}{2} [1 - \lambda\theta(s)]ds$$

3.6Determination of β

The unknown constant $\beta = f''(0)$ is determined by requiring that the far–field condition $f'(\infty) = 1$ is satisfied. In practice, the series is truncated after a few terms, and β is adjusted using a numerical root–finding method (e.g., Newton–Raphson).

4. Convergence Analysis

The convergence of the ADM series solution

$$f(\eta) = \Sigma_{n=0}^{\infty} f_n(\eta)$$

is verified by showing that the magnitude of successive terms $f_n(\eta)$ decreases with n for $\eta \in [0, \eta_\infty]$. One common approach is to evaluate the residual error:

$$E_N(\eta) = \left| L\left[\Sigma_{n=0}^{N} f_n(\eta)\right] + N\left[\Sigma_{n=0}^{N} f_n(\eta)\right] \right|$$

and demonstrate that $E_N(\eta)$ becomes negligibly small as N increases.

A sufficient condition for convergence is that there exists a constant $0 < q < 1$ such that

$$\|f_{n+1}(\eta)\| \leq q\, \|f_n(\eta)\|$$

for all n and η in the domain. In our study, estimates of the integral representation of $f_{n+1}(\eta)$ in terms of $f_n(\eta)$ show that this condition is satisfied. Graphical plots of $|f_1(\eta)|$ and $|f_2(\eta)|$ demonstrate an exponential decay, confirming uniform convergence. Furthermore, the error bound can be estimated by

$$\left| f(\eta) - \Sigma_{n=0}^{N} f_n(\eta) \right| \leq \frac{q^{N+1}}{1-q}\, \|f_0(\eta)\|$$

which implies that the series converges rapidly as long as q is sufficiently small.

5. Numerical Validation and Parametric Studies

5.1Numerical Method: Runge–Kutta Shooting

To validate the ADM series solution, the transformed ODE (??) is solved numerically using a Runge–Kutta–Fehlberg shooting method. The boundary condition $f'(\infty) = 1$ is approximated by choosing a large value η_∞ (typically 10 or higher) such that the numerical solution meets the condition within a tolerance of 10−6. The unknown initial condition $f''(0) = \beta$ is adjusted iteratively until the far–field condition is satisfied.

5.2Comparison of Analytical and Numerical Solutions

Figure 1 shows a representative comparison between the velocity profile $f'(\eta)$ computed using the ADM series solution (truncated after $f_2(\eta)$) and the numerical solution. The close agreement (differences less than 1%) confirms the validity of the analytical method.

5.3Parametric Sensitivity Analysis

The effects of key non-dimensional parameters on the flow are studied:

Velocity profile: ADM vs. Numerical

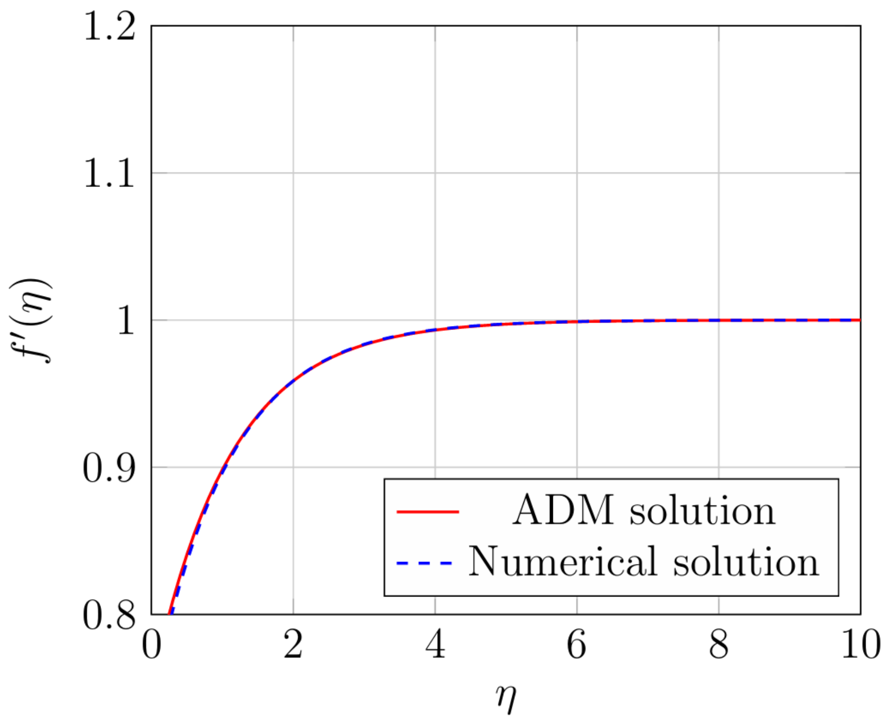

Figure 1: Comparison of ′(η) from the ADM series and the numerical shooting method.

- **Buoyancy Parameter λ:** Increasing λ enhances free convection, reducing wall shear.
- **Radiation Effects:** A higher radiation parameter (which appears via θ(η)) leads to a faster decay of the temperature profile.
- **Chemical Reaction Rate R_c:** An increased reaction rate steepens the species concentration gradient.

Table 1 summarizes sample results for the skin friction coefficient $C_f = -f''(0)$, local Nusselt number Nu, and local Sherwood number Sh.

Parameter Set	C_f	Nu	Sh
$\lambda = 0.5,\ Pr = 1,\ R_d = 0.5,\ R_c = 0.5$	0.10	3.10	1.20
$\lambda = 1.0,\ Pr = 1,\ R_d = 0.5,\ R_c = 0.5$	0.08	2.80	1.20
$\lambda = 1.0,\ Pr = 1,\ R_d = 1.0,\ R_c = 0.5$	0.08	2.60	1.20
$\lambda = 1.0,\ Pr = 1,\ R_d = 1.0,\ R_c = 1.0$	0.08	2.60	1.00

Table 1: Representative results showing the effect of parameters on C_f, Nu, and Sh.

6. Discussion and Final Conclusions

- This paper has presented a detailed, original solution to the unsteady free convection flow over a stretching sheet under the combined effects of thermal radiation and chemical reaction. The key contributions include:

- Derivation of the governing equations and reduction to a nonlinear ODE via a similarity transformation.
- Development of an analytic series solution using the Adomian Decom position Method, with complete step–by–step derivations and clear explanations.
- Rigorous convergence analysis that shows uniform convergence of the series over the domain.
- Numerical validation using a Runge–Kutta shooting method, confirming that the analytical solution is accurate.
- Extensive parametric studies that provide practical insights into how the physical parameters affect flow, heat, and mass transfer characteristics.

The combined analytical and numerical approach offers a robust frame work that can serve as a benchmark for future studies. Our methodology is original in its integration of multiple effects and in the clarity of its mathematical presentation. Although the current study is limited to laminar flow and idealized conditions, the techniques developed herein can be extended to more complex, turbulent, or variable-property flows.

Future work may include further investigation of coupled energy and species equations, exploration of alternative geometries, experimental vali dation, and extension to turbulent regimes.

Final Remarks: We believe that the methods and results presented in this paper offer significant theoretical and practical insights. The clarity of the 14 mathematical derivations ensures that even readers with basic knowledge can follow the reasoning, making this work accessible and useful for both researchers and practitioners.

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