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Transient Analysis of a Queueing System Incorporating the Effects of Feedback, Randomly Changing States and Catastrophes

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Abstract: In this paper, a limited capacity queueing system incorporating the effects of feedback, randomly changing states and catastrophes is studied. The effect of two randomly changing states are taken to be a function of the number of customers present in the system. We undertake the transient analysis of a limited capacity queueing system with two randomly changing states in the presence of feedback and catastrophes. Transient solution of the queueing model is obtained by using the probability generating function technique. Some interesting particular cases of the queueing model with (without) feedback and catastrophes are obtained. Measures of effectiveness and steady state solutions of the model are also discussed.

Keywords: *Transient analysis, Feedback, Catastrophe, Randomly changing states, Probability generating function.*

1. Introduction:

In queueing theory, the M/M/1 queueing system has been the object of systematic and through investigations by many authors. In recent years, the attention has been focused to study the queueing systems on certain extensions that include the effect of feedback and catastrophes. These two factors consists of adding to the standard assumptions of the existing models. The catastrophes occur at the service- facility as a Poisson process with rate ξ. Whenever a catastrophe occurs at the system, all the customers there are destroyed immediately, the server gets inactivated momentarily, and the server is ready for service when a new arrival occurs. The concept of catastrophe played a very important role in many areas of science, technology and management.

The feedback in queueing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, the customer retries for service. This usually happens because of non-satisfactory quality of service. Also, the another factor of randomly changing states, i.e. the change in the two randomly states affects the state of the queueing system. In other words, the state of the queueing system is a function of randomly changing factors. This paper is the generalization of our previous work [Kumar, D. (9)] in which we do not consider the concept of customers feedback.

Kumar and Arivudainambi [3] obtained the transient solution of M/M/1 queueing model with the possibility of catastrophes at the service station. Jain and Kanethia [10], Goel [8] studied the transient analysis of a queue with environmental and catastrophic effects. Several authors have investigated queueing systems subject to feedback. Takacs [14], in his interesting paper has introduced a queue with feedback. D'Avignon and Disney [4] studied single server queues with state dependent feedback. Thangaraj and Vanitha [15], obtained transient solution of M/M/1 feedback queue with catastrophes using continued fractions and the steady-state solution. Kumar and Sharma [13]

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studied a M/M/1 Feedback queuing model with retention of reneged customers and Balking.

In this paper, we undertake the transient analysis of a queueing system incorporating the effects of feedback, catastrophes and randomly changing states in order to obtain some analytical results. In section 2, we have made the assumptions and definitions of the model. The detailed analysis of the main model is done in section 3 and some particular cases are obtained in section 4. In section 5 & 6, we have obtained the steady-state result and mean queue length. Also, the application of the model is discussed.

2. Assumptions and Definitions:

(i) The customers arrive in the system one by one in accordance with a Poisson process at a single service station. The arrival pattern is non-homogeneous, i.e. there may exists two arrival rates, namely λ_1 and 0 of which only one is operative at any instant.

(ii) The customers are served one by one at the single channel. The service time is exponentially distributed. Further, corresponding to arrival rate λ_1 the Poisson service rate is μ_1 and the service rate corresponding to the arrival rate 0 is μ_2. The state of the system when operating with arrival rate λ_1 and service rate μ_1 is designated as E whereas the other with arrival rate 0 and service rate μ_2 is designated as F.

(iii) In state E after obtaining a service, an unsatisfied customer may rejoin the queue for receiving another service with probability 1-p (=q), referred to as "feedback" or they can choose to leave the system permanently with probability p, p+q=1. There is no feedback in state F of the system.

(iv) The Poisson rate d_n at which the system goes from environmental state E to F tends to decrease or increase whereas at the same time the Poisson rate b_n at which the system moves from environmental state F to E tends to increase or decrease according as the numbers in the queue (say n) increase or decrease from some fixed number (say N). We therefore define,

$$d_n = \beta\left[1+\varepsilon'(N-n)\right] \text{ with } n \le N + \frac{1}{\varepsilon'}$$

$$\text{and } \quad 0 \le n \le N + \frac{1}{\varepsilon'} \le M$$

Also

$$b_n = \alpha\left[1+\varepsilon(n-N)\right] \text{ with } n \ge N - \frac{1}{\varepsilon}$$

$$\text{and } \quad 0 \le N - \frac{1}{\varepsilon} \le n \le M$$

Where M denotes the size of the waiting space and ε, ε' are positive numbers such that $\varepsilon \ge \frac{1}{N}$ and $\varepsilon' \ge \frac{1}{M-N}$. These restrictions on M also are necessary to avoid the negative values of d_n and b_n. When n=N or ε=0, b_n gives the normal rate as α and when n=N or $\varepsilon = \varepsilon'$=0, d_n and b_n gives the normal rates as β and α.

(iv) When the system is not empty, catastrophes occur according to a Poisson process with rate ξ. The effect of each catastrophe is to make the queue instantly empty. Simultaneously, the system becomes ready to accept the new customers.

(v) The queue discipline is first- come-first-served.

(vi) The capacity of the system is limited to M. i.e., if at any instant there are M units in the queue then the units arriving at that instant will not be permitted to join the queue, it will be considered lost for the system.

Define,

$P_n(t)$ = Joint probability that at time t the system is in state E and n units are in the queue, including the one in service.

$Q_n(t)$ = Joint probability that at time t the system is in state F and n units are in the queue, including the one in service.

$R_n(t)$ = The probability that at time t there are n units in the queue, including the one in service.

Obviously,

$$R_n(t) = P_n(t) + Q_n(t)$$

Let us reckon time t from an instant when there are zero customers in the queue and the system is in the environmental state E so that the initial conditions associated with $P_n(t)$ and $Q_n(t)$ becomes,

$$P_n(0) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$Q_n(0) = 0 \; ; \quad \text{for all n.}$$

3. Formulation of Model and Analysis (Time Dependent Solution):

The differential-difference equations governing the system are:

$$\frac{d}{dt}P_0(t) = -(\lambda_1 + d_0 + \xi)P_0(t) + \mu_1 p P_1(t) + b_0 Q_0(t) + \xi\sum_{n=0}^{M} P_n(t)\; ; \quad n = 0 \quad \text{..... (1)}$$

$$\frac{d}{dt}P_n(t) = -(\lambda_1 + \mu_1 p + d_n + \xi)P_n(t) + \mu_1 p P_{n+1}(t) + \lambda_1 P_{n-1}(t) + b_n Q_n(t)\; ;$$

$$0 < n < M \quad \text{..... (2)}$$

$$\frac{d}{dt}P_M(t) = -(\mu_1 p + d_M + \xi)P_M(t) + \lambda_1 P_{M-1}(t) + b_M Q_M(t)\; ; \quad n = M \quad \text{.... (3)}$$

$$\frac{d}{dt}Q_0(t) = -(b_0 + \xi)Q_0(t) + \mu_2 Q_1(t) + d_0 P_0(t) + \xi\sum_{n=0}^{M} Q_n(t)\; ; \quad n = 0 \quad \text{.... (4)}$$

$$\frac{d}{dt}Q_n(t) = -(\mu_2 + b_n + \xi)\,Q_n(t) + \mu_2\,Q_{n+1}(t) + d_n P_n(t)\; ; \quad 0 < n < M \quad \text{.... (5)}$$

$$\frac{d}{dt}Q_M(t) = -(\mu_2 + b_M + \xi)Q_M(t) + d_M P_M(t)\; ; \quad n = M \quad \text{.... (6)}$$

Define, the Laplace Transform as

$$\text{L.T. } [f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt = \bar{f}(s) \quad \text{.....(7)}$$

Now, taking the Laplace transforms of equations (1)–(6) and using the initial conditions, we get

$$(s + \lambda_1 + d_0 + \xi)\bar{P}_0(s) - 1 = \mu_1 p\,\bar{P}_1(s) + b_0\,\bar{Q}_0(s) + \xi\sum_{n=0}^{M}\bar{P}_n(s) \quad \text{.... (8)}$$

$$(s + \lambda_1 + \mu_1 p + d_n + \xi)\bar{P}_n(s) = \mu_1 p\bar{P}_{n+1}(s) + \lambda_1\,\bar{P}_{n-1}(s) + b_n\,\bar{Q}_n(t) \quad \text{.... (9)}$$

$$(s + \mu_1 p + d_M + \xi)\bar{P}_M(s) = \lambda_1\,\bar{P}_{M-1}(s) + b_M\,\bar{Q}_M(s) \quad \text{.... (10)}$$

$$\left(s+b_0+\xi\right)\overline{Q}_0(s)=\mu_2\,\overline{Q}_1(s)+d_0\,\overline{P}_0(s)+\xi\sum_{n=0}^{M}\overline{Q}_n(s) \quad \text{.... (11)}$$

$$\left(s+\mu_2+b_n+\xi\right)\overline{Q}_n(s)=\mu_2\,\overline{Q}_{n+1}(s)+d_n\,\overline{P}_n(s) \quad \text{.... (12)}$$

$$\left(s+\mu_2+b_M+\xi\right)\overline{Q}_M(s)=d_M\,\overline{P}_M(s) \quad \text{.... (13)}$$

Define, the probability generating functions

$$P(z,s)=\sum_{n=0}^{M}\overline{P}_n(s)z^n \quad \text{.... (14)}$$

$$Q(z,s)=\sum_{n=0}^{M}\overline{Q}_n(s)z^n \quad \text{.... (15)}$$

$$R(z,s)=\sum_{n=0}^{M}\overline{R}_n(s)z^n \quad \text{.... (16)}$$

where

$$\overline{R}_n(s)=\overline{P}_n(s)+\overline{Q}_n(s)$$

Multiplying equations (8)–(10) by the suitable powers of z, summing over all n and using equations (14)–(16), we have.

$$\beta\varepsilon' z^2 P'(z,s)+\alpha\varepsilon z^2\,Q'(z,s)+\left[\lambda_1 z^2-z\left\{s+\lambda_1+\mu_1 p+\xi+\beta(1+\varepsilon' N)\right\}+\mu_1 p\right]P(z,s)$$

$$+\alpha(1-\varepsilon N)z\,Q(z,s)=\lambda_1 z^{M+1}(z-1)\overline{P}_M(s)+\mu_1 p(1-z)\overline{P}_0(s)-z-\xi z\sum_{n=0}^{M}\overline{P}_n(s) \quad \text{.... (17)}$$

Similarly, from equations (11)–(13) and using (14)–(16), we have

$$\beta\varepsilon' z^2 P'(z,s)+\alpha\varepsilon z^2\,Q'(z,s)-\beta(1+\varepsilon' N)z\,P(z,s)+\left[z\left\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\right\}-\mu_2\right]Q(z,s)$$

$$=\mu_2(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s) \quad \text{.... (18)}$$

Subtracting equation (18) from (17), we have.

$$\left[\lambda_1 z^2-z\left(s+\lambda_1+\mu_1 p+\xi\right)+\mu_1 p\right]P(z,s)+\left[\mu_2-z\left(s+\mu_2+\xi\right)\right]Q(z,s)=\lambda_1 z^{M+1}(z-1)\overline{P}_M(s)$$

$$+\mu_1 p(1-z)\overline{P}_0(s)-\mu_2(z-1)\overline{Q}_0(s)-z-\xi z\sum_{n=0}^{M}\overline{P}_n(s)-\xi z\sum_{n=0}^{M}\overline{Q}_n(s) \quad \text{.... (19)}$$

Differentiating equation (19) with respect to z, we have

$$\left[\lambda_1 z^2-z\left(s+\lambda_1+\mu_1 p+\xi\right)+\mu_1 p\right]P'(z,s)+\left[2\lambda_1 z-\left(s+\lambda_1+\mu_1 p+\xi\right)\right]P(z,s)$$

$$+\left[\mu_2-z\left(s+\mu_2+\xi\right)\right]Q'(z,s)-\left(s+\mu_2+\xi\right)Q(z,s)=\lambda_1 z^M\left[(M+2)z-(M+1)\right]\overline{P}_M(s)$$

$$-\mu_1 p\overline{P}_0(s)-\mu_2\,\overline{Q}_0(s)-1-\xi\sum_{n=0}^{M}\overline{P}_n(s)-\xi\sum_{n=0}^{M}\overline{Q}_n(s) \quad \text{.... (20)}$$

Eliminating Q'(z,s) and Q(z,s) from equations (18), (19) and (20), we arrive at a computationally convenient equation.

$$P'(z,s)+\frac{\eta_1(z)}{\eta_2(z)}P(z,s)=\frac{1}{\eta_2(z)}\left[z_1+z_2\overline{Q}_0(s)+z_3\overline{P}_0(s)+z_4\,\overline{P}_M(s)+z_5\sum_{n=0}^{M}\overline{P}_n(s)+z_6\sum_{n=0}^{M}\overline{Q}_n(s)\right] \quad \text{.... (21)}$$

where

$$\eta_1(z)=a_1z^4+a_2z^3+a_3z^2+a_4z+a_5$$

$$\eta_2(z)=z^2\left[\mu_2-z(s+\mu_2+\xi)\right]\left[a_6z^2+a_7z+a_8\right]$$

$$\frac{\eta_1(z)}{\eta_2(z)}=\frac{A}{\mu_2-z(s+\mu_2+\xi)}+(B+C/z)\frac{1}{z}+\frac{D(2a_6z+a_7)}{a_6(a_6z^2+a_7z+a_8)}$$

$$+\frac{(E-a_7D/2a_6)}{a_6\left[\left(z+\dfrac{a_7}{2a_6}\right)^2-\left\{\left(\dfrac{1}{2}a_7/a_6\right)^2-\left(\dfrac{a_8}{a_6}\right)\right\}\right]}$$

$$C=\frac{a_5}{\mu_2a_8}$$

$$B=\left[a_4-\frac{a_5}{\mu_2a_8}\{\mu_2a_7-(s+\mu_2+\xi)a_8\}\right]\frac{1}{\mu_2a_8}$$

$$A=\begin{vmatrix}(a_3-b_1B-b_2c) & 0 & \mu_2\\ (a_2-b_2B-b_3c) & \mu_2 & b_4\\ (a_1-b_3B) & b_4 & 0\end{vmatrix}\frac{1}{\Delta}$$

$$D=\begin{vmatrix}a_8 & \mu_2 & (a_3-b_1B-b_2C)\\ a_7 & b_4 & (a_2-b_2B-b_3C)\\ a_6 & 0 & (a_1-b_3B)\end{vmatrix}\frac{1}{\Delta}$$

$$E=\begin{vmatrix}a_8 & 0 & (a_3-b_1B-b_2C)\\ a_7 & \mu_2 & (a_2-b_2B-b_3C)\\ a_6 & b_4 & (a_1-b_3B)\end{vmatrix}\frac{1}{\Delta}$$

$$\Delta=\begin{vmatrix}a_8 & 0 & \mu_2\\ a_7 & \mu_2 & b_4\\ a_6 & b_4 & 0\end{vmatrix}$$

$b_1 = \mu_2\, a_7 - a_8\, (s+\mu_2+\xi)$

$b_2 = \mu_2\, a_6 - a_7\, (s+\mu_2+\xi)$

$b_3 = -\, a_6\, (s+\mu_2+\xi)$

$b_4 = -\, (s+\mu_2+\xi)$

$$a_1=\lambda_1(s+\mu_2+\xi)\left[\alpha\varepsilon+\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}\right]$$

$$a_2 = -\Big[(s+\mu_2+\xi)\{\beta(1+\varepsilon' N)(s+\mu_2+\xi)+\lambda_1\mu_2\} + \{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}.$$
$$\{\lambda_1\mu_2 + (s+\mu_2+\xi)(s+\lambda_1+\mu_1 p+\xi)\} + 2\lambda_1\varepsilon\,\mu_2\,\alpha \Big]$$

$$a_3 = \big[(s+\mu_2+\xi)\{2\beta\mu_2(1+\varepsilon' N) - \alpha\varepsilon\mu_1 p\} + \mu_2(s+\lambda_1+\mu_1 p+\xi)$$
$$\{\alpha\varepsilon + (s+\mu_2+\xi)\} + \{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}$$
$$\{\mu_1 p(s+\mu_2+\xi) + \mu_2(s+\lambda_1+\mu_1 p+\xi)\} + (\mu_2 z)^2\lambda_1 \big]$$

$$a_4 = -\big[\mu_2^2\{\beta(1+\varepsilon' N) + (s+\lambda_1+\mu_1 p+\xi)\} + \mu_1 p\mu_2$$
$$\{s+\mu_2+\xi+\alpha(1-\varepsilon N) + (s+\mu_2+\xi)\} \big]$$

$$a_5 = \mu_1 p\mu_2^2$$

$$a_6 = -\alpha\varepsilon\lambda_1^2$$

$$a_7 = \alpha\varepsilon\,(s+\lambda_1+\mu_1 p+\xi) - \beta\varepsilon'\,(s+\mu_2+\xi)$$

$$a_8 = (\beta\,\varepsilon'\,\mu_2 - \alpha\,\varepsilon\,\mu_1 p)$$

$$z_1 = \big[-z^3(s+\mu_2+\xi)[\alpha\varepsilon + \{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}]$$
$$+ z^2\mu_2\,[2(s+\mu_2+\xi) + \alpha(1-\varepsilon N) + \alpha\varepsilon] - z\mu_2^2$$

$$z_2 = z(z-1)\big[\,\mu_2\{2(s+\mu_2+\xi) + \alpha(1-\varepsilon N)\} - z(s+\mu_2+\xi)\cdot$$
$$\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}\big] - \mu_2(z-1)\big[\mu_2^2 - \{\mu_2 - z(s+\mu_2+\xi)\}^2\big]$$
$$+\mu_2\,\varepsilon\,\alpha z^2\{\mu_2 - z(s+\mu_2+\xi)\}$$

$$z_3 = \big[\mu_1 p\mu_2\{2(s+\mu_2+\xi) + \alpha(1-\varepsilon N)\}z(z-1) + \mu_1 p(s+\mu_2+\xi)z^2(1-z).$$
$$\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\} + (1-z)\mu_1 p\mu_2^2 + \mu_1 p z^2\alpha\varepsilon$$
$$\cdot\{\mu_2 - (s+\mu_2+\xi)z\}\big]$$

$$z_4 = z^M\big[z^3(z-1)\lambda_1(s+\mu_2+\xi)\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}$$
$$-z^2(z-1)\lambda_1\mu_2\{2(s+\mu_2+\xi) + \alpha(1-\varepsilon N)\} +$$
$$z(z-1)\lambda_1\mu_2^2 - \alpha\varepsilon z^2\lambda_1\{z(M+2) - (M+1)\}\{\mu_2 - z(s+\mu_2+\xi)\}\big]$$

$$z_5 = z^3\xi(s+\mu_2+\xi)\,[(s+\mu_2+\xi) - \{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}] +$$
$$z^2\mu_2\,\xi\,[\alpha\varepsilon + \{s+\mu_2+\xi+\alpha(1-\varepsilon N)\} - (s+\mu_2+\xi)]$$

$$z_6 = -z^3\xi(s+\mu_2+\xi)\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\} +$$
$$z^2\mu_2\xi\,[\alpha\varepsilon + (s+\mu_2+\xi) + \{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}] - z{\mu_2}^2\xi$$

On solving equation (21), we have

$$P(z,s)=\frac{L_1(z)+L_2(z)\overline{Q}_0(s)+L_3(z)\overline{P}_0(s)+L_4(z)\overline{P}_M(s)+L_5(z)\sum_{n=0}^{M}\overline{P}_n(s)+L_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{L(z)} \quad \ldots(22)$$

where

$$L(z)=\left[\mu_2-z(s+\mu_2+\xi)\right]^{-A/(s+\mu_2+\xi)}\left[\frac{X(z)-a}{X(z)+a}\right]^{E'}$$

$$z^B\cdot\left(a_6z^2+a_7z+a_8\right)^{D'}\exp^{(-C/2)}$$

$$D'=\frac{D}{2a_6}$$

$$E'=\left(E-\frac{a_7}{2a_6}D\right)\frac{1}{2a_6}$$

$$X(z)=z+\frac{a_7}{2a_6}$$

$$a=\left[\left(\frac{1}{2}\frac{a_7}{a_6}\right)^2-\left(\frac{a_8}{a_6}\right)\right]^{½}$$

$$L_j(z)=\int_0^Z\frac{z_j}{\eta_2(z)}L(z)\,dz\,; \qquad j=1, 2, 3, 4, 5, 6.$$

Now, from equations (19) and (22), we have

$$Q(z,s)=\frac{L_7(z)+\overline{Q}_0(s)L_8(z)+\overline{P}_0(s)L_9(z)+\overline{P}_M(s)L_{10}(z)+\sum_{n=0}^{M}\overline{P}_n(s)L_{11}(z)+\sum_{n=0}^{M}\overline{Q}_n(s)L_{12}(z)}{B(z)L(z)} \quad \ldots(23)$$

where

$$L_7(z)=L_1(z)\,g(z)-z\,L(z)$$

$$L_8(z)=L_2(z)\,g(z)-\mu_2(z-1)\,L(z)$$

$$L_9(z)=L_3(z)\;g(z)-\mu_1p(z-1)\,L(z)$$

$$L_{10}(z)=L_4(z)\;g(z)+\lambda_1z^{M+1}(z-1)\,L(z)$$

$$L_{11}(z)=L_5(z)\;g(z)-\xi z\;L(z)$$

$$L_{12}(z)=L_6(z)\;g(z)-\xi z\;L(z)$$

$$g\,(z)=\left[z\,(s+\lambda_1+\mu_1p+\xi)-\mu_1p-\lambda_1z^2\,\right]$$

$$B(z)=\left[\mu_2-z(s+\mu_2+\xi)\right]$$

Adding equations (22) and (23), we have

$$R(z,s)=\frac{C_1(z)+C_2(z)\overline{Q}_0(s)+C_3(z)\overline{P}_0(s)+C_4(z)\overline{P}_M(s)+C_5(z)\sum_{n=0}^{M}\overline{P}_n(s)+C_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B(z)\,L(z)} \quad .... (24)$$

where

$C_i(z) = B(z)\,L_i(z) + L_{i+6}(z)$; i=1, 2, 3, 4, 5, 6.

Since,

$$R(1,s)=\sum_{n=0}^{M}\overline{R}_n(s)=\frac{1}{s} \quad .... (25)$$

Thus equation (24) for z=1, gives

$$R(1,s)=\frac{1}{s}=\lim_{z\to 1}R(z,s) \quad .... (26)$$

$$P(0,s)=\overline{P}_0(s)=\lim_{z\to 0}P(z,s) \quad .... (27)$$

And $$Q(0,s)=\overline{Q}_0(s)=\lim_{z\to 0}Q(z,s) \quad .... (28)$$

The equations (26), (27) and (28) on solution gives the values of $\overline{P}_0(s)$, $\overline{Q}_0(s)$, $\overline{P}_M(s)$, $\sum_{n=0}^{M}\overline{P}_n(s)$ and $\sum_{n=0}^{M}\overline{Q}_n(s)$.

Again, we have from equations (22) and (23) on setting z=1 and $\overline{P}_0(s) = P_0$, $\overline{Q}_0(s) = Q_0$, $\overline{P}_M(s) = P_M$, $\sum_{n=0}^{M}\overline{P}_n(s)=P_n$ and $\sum_{n=0}^{M}\overline{Q}_n(s)=Q_n$

$$P(1,s)=\frac{L_1(1)+L_2(1)Q_0+L_3(1)P_0+L_4(1)P_M+L_5(1)\sum_{n=0}^{M}P_n+L_6(1)\sum_{n=0}^{M}Q_n}{L(1)} \quad .... (29)$$

$$Q(1,s)=\frac{L_7(1)+L_8(1)Q_0+L_9(1)P_0+L_{10}(1)P_M+L_{11}(1)\sum_{n=0}^{M}P_n+L_{12}(1)\sum_{n=0}^{M}Q_n}{B(1)\,L(1)} \quad .... (30)$$

These on inversions give the respective probabilities for the system to be in the randomly changing states E and F.

4. Particular Cases:

Case I Setting n=N or $\varepsilon=0$ in equations (22) and (23), (i.e., when the rate of change of randomly changing state from F to E is constant), we have

$$P(z,s)=\frac{L_1'(z)+L_2'(z)\overline{Q}_0(s)+L_3'(z)\overline{P}_0(s)+L_4'(z)\overline{P}_M(s)+L_5'(z)\sum_{n=0}^{M}\overline{P}_n(s)+L_6'(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{L'(z)} \quad ...(31)$$

$$Q(z,s)=\frac{L_7'(z)+L_8'(z)\overline{Q}_0(s)+L_9'(z)\overline{P}_0(s)+L_{10}'(z)\overline{P}_M(s)+L_{11}'(z)\sum_{n=0}^{M}\overline{P}_n(s)+L_{12}'(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B'(z)\,L'(z)} \quad ...(32)$$

where

$$L_i'(z)=L_i(z)\big|_{\varepsilon=0} \quad ; \; i=1, 2, 3, \ldots . 12.$$

$$L'(z)=L(z)\big|_{\varepsilon=0}$$

$$B'(z)=B(z)\big|_{\varepsilon=0}$$

On adding equations (31) and (32), we have.

$$R(z,s)=\frac{C_1'(z)+C_2'(z)\overline{Q}_0(s)+C_3'(z)\overline{P}_0(s)+C_4'(z)\overline{P}_M(s)+C_5'(z)\sum_{n=0}^{M}\overline{P}_n(s)+C_6'(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B'(z)\,L'(z)} \quad \ldots (33)$$

where

$$C_i'(z)=B'(z)L_i'(z)+L_{i+6}'(z) \; ; \; i=1, 2, 3, 4, 5, 6.$$

The unknown quantities $\overline{Q}_0(s)$, $\overline{P}_0(s)$, $\overline{P}_M(s)$, $\sum_{n=0}^{M}\overline{P}_n(s)$ and $\sum_{n=0}^{M}\overline{Q}_n(s)$ can be evaluated as before.

Case II Setting $\varepsilon=\varepsilon'=0$ or n=N in equation (17) and (18), (i.e. when the rates of interchange of randomly changing states from E to F and F to E is constant), we have

$$X_1(z)P(z,s)+X_2(z)Q(z,s)+X_3(z)=0 \quad \ldots . (34)$$

$$X_4(z)P(z,s)+X_5(z)Q(z,s)+X_6(z)=0 \quad \ldots . (35)$$

where

$$X_1(z)=-\left[\lambda_1 z^2-z\,(s+\lambda_1+\mu_1 p+\beta+\xi)+\mu_1 p\right]$$

$$X_2(z)=-\alpha z$$

$$X_3(z)=-\left[\mu_1 p(z-1)\,\overline{P}_0(s)+\lambda_1 z^{M+1}(1-z)\overline{P}_M(s)+z+\xi z\sum_{n=0}^{M}\overline{P}_n(s)\right]$$

$$X_4(z)=\beta z$$

$$X_5(z)=\left[\mu_2-z(s+\mu_2+\alpha+\xi)\right]$$

$$X_6(z)=\left[\mu_2(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)\right]$$

From equations (34) and (35), we have

$$P(z,s)=\frac{X_2(z)X_6(z)-X_3(z)X_5(z)}{X_1(z)X_5(z)-X_2(z)X_4(z)} \quad \ldots . (36)$$

$$Q(z,s)=\frac{X_4(z)X_3(z)-X_1(z)X_6(z)}{X_1(z)X_5(z)-X_2(z)X_4(z)} \quad \ldots . (37)$$

Thus, we have

$$R(z,s)=\frac{\begin{array}{l}\mu_2(z-1)[X_2(z)-X_1(z)]\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)[X_2(z)-X_1(z)]+\mu_1 p(1-z)\\ {[X_4(z)-X_5(z)]}\overline{P}_0(s)+\lambda_1 z^{M+1}[X_5(z)-X_4(z)](1-z)\overline{P}_M(s)+\\ z[X_5(z)-X_4(z)]+\xi z\sum_{n=0}^{M}\overline{P}_n(s)[X_5(z)-X_4(z)]\end{array}}{-z^2 s^2+sX_7(z)+(1-z)X_8(z)-z^2\xi(\alpha+\beta+\xi)} \quad \text{.... (38)}$$

where

$$X_7(z)=\lambda_1 z^3-z^2(\lambda_1+\mu_1 p+\mu_2+\alpha+\beta+2\xi)+z(\mu_1 p+\mu_2)$$

$$X_8(z)=-z^2\lambda_1(\alpha+\mu_2+\xi)+z[\alpha\mu_1 p+\mu_2(\lambda_1+\mu_1 p+\beta+\xi)]-\mu_1 p\mu_2 .$$

And $P(1,s)=\sum_{n=0}^{M}\overline{P}_n(s)=\dfrac{s+\alpha+\xi}{s(s+\alpha+\beta+\xi)}$

$$Q(1,s)=\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{\beta}{s(s+\alpha+\beta+\xi)}$$

Relation (38) is a polynomial in z and exists for all values of z, including the three zeros of the denominator. Hence $\overline{P}_0(s)$, $\overline{Q}_0(s)$ and $\overline{P}_M(s)$ are obtained by setting the numerator equal to zero and substituting the three zeros, α_1, α_2 and α_3 (say) of the denominator (at each of which the numerator must vanish).

Now letting $\alpha\to\infty$, $\beta\to 0$ and setting $\mu_1=\mu_2=\mu$ (say) in relation (38), we have

$$r(z,s)=\frac{(1-z)\mu\overline{R}_0(s)-(1-z)\lambda_1 z^{M+1}\overline{P}_M(s)-z-\xi z/s}{\lambda_1 z^2-z(s+\lambda_1+\mu+\xi)+\mu} \quad \text{.... (39)}$$

where

$$\overline{R}_0(s)=\overline{P}_0(s)+\overline{Q}_0(s)$$

$$r(z,s)=\lim_{\beta\to 0}\left[\lim_{\alpha\to\infty}R(z,s)\right]$$

Relation (39) is a polynomial in z and exists for all values of z, including the two zeros of the denominator. Hence, $\overline{R}_0(s)$ and $\overline{P}_M(s)$ can be evaluated as before.

Case III Putting $\varepsilon=\varepsilon'=1$, N=1 in equation (24), (i.e. when $d_n=\beta n$ and $b_n=\alpha n$), we have.

$$R(z,s)=\frac{C_1''(z)+\overline{Q}_0(s)C_2''(z)+\overline{P}_0(s)C_3''(z)+\overline{P}_M(s)C_4''(z)+\sum_{n=0}^{M}\overline{P}_n(s)C_5''(z)+\sum_{n=0}^{M}\overline{Q}_n(s)C_6''(z)}{B(z)L''(z)} \text{... (40)}$$

where

$$L''(z)=L(z)\big|_{\varepsilon=\varepsilon'=1,\,N=1}$$

$$C_i''(z)=C_i(z)\big|_{\varepsilon=\varepsilon'=1,\,N=1};\quad i=1, 2, 3, 4, 5, 6$$

Case IV: In relation (38), if p=1, i.e. when there are no feedback customers. The model reduces to one which is studied by Kumar, D. [9].

5. Steady State Results:

This can at once be obtained by the well-known property of the Laplace transform given below:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,\bar{f}(s),$$ If the limit on the left hand side exists.

Thus if

$$R(z)=\sum_{n=0}^{M} R_n\ z^n$$

Then

$$R(z)=\lim_{s\to 0} s\ R\ (z,s)$$

By using this property, we have from equation (24) for the steady state

$$R(z)=\frac{N_1(z)Q_0 + N_2(z)P_0 + N_3(z)P_M + N_4(z)\sum_{n=0}^{M}P_n + N_5(z)\sum_{n=0}^{M}Q_n + N}{B^*(z)L^*(z)} \qquad \text{.... (41)}$$

where

$$R_n = \lim_{s\to 0} s\ \bar{R}_n(s)$$

$$N_i(z) = B(z)L^*_{i+1}(z) + L^*_{i+7}(z)\Big|_{s=0}\ ;\ i=1,\ 2,\ 3,\ 4,\ 5$$

$$B^*(z)=B(z)\Big|_{s=0}$$

$$L^*(z)=L(z)\Big|_{s=0}$$

$$L_i^*(z)=\int \frac{z_i}{\eta_2(z)} L(z)\,dz\ \ ;\ i=1,\ 2,\ 3,\ 4,\ 5,\ 6$$

$$L_8^*(z)=L_2^*(z)\,g(z)-\mu_2(z-1)L(z)$$

$$L_9^*(z)=L_3^*(z)\,g(z)-\mu_1(z-1)L(z)$$

$$L_{10}^*(z)=L_4^*(z)\,g(z)+\lambda_1 z^{M+1}(z-1)L(z)$$

$$L_{11}^*(z)=L_5^*(z)\,g(z)-\xi z\,L(z)$$

$$L_{12}^*(z)=L_6^*(z)\,g(z)-\xi z\,L(z)$$

and N = The constant of integration.

The unknown quantities P_0, Q_0, P_M, $\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}Q_n$ can be evaluated as before.

Particular cases:

Case I Relation (33), on applying the theory of Laplace transform, we have

$$R(z)=\frac{Q_0 N_1'(z)+N_2'(z)P_0+N_3'(z)P_M+N_4'(z)\sum_{n=0}^{M}P_n+N_5'(z)\sum_{n=0}^{M}Q_n+N'}{T_1(z)\,T_2(z)} \quad \text{.... (42)}$$

where,

$$N_i'(z)=B'(z)L_{i+1}^{**}(z)+L_{i+7}^{**}(z)\Big|_{s=0} \quad ;\ i=1, 2, 3, 4, 5.$$

$$T_1(z)=B'(z)\big|_{s=0}$$

$$L_j^{**}(z)=\int\left[\frac{z_j}{\eta_2(z)}L(z)\right]_{\in=0} dz \quad ;\ j=2, 3, 4, 5, 6.$$

$$L_8^{**}(z)=L_2^{**}(z)\ g(z)-\mu_2(z-1)L'(z)$$

$$L_9^{**}(z)=L_3^{**}(z)\ g(z)-\mu_1(z-1)L'(z)$$

$$L_{10}^{**}(z)=L_4^{**}(z)\ g(z)+\lambda_1 z^{M+1}(z-1)L'(z)$$

$$L_{11}^{**}(z)=L_5^{**}(z)\ g(z)-\xi z\,L'(z)$$

$$L_{12}^{**}(z)=L_6^{**}(z)\ g(z)-\xi z\,L'(z)$$

N' = the constant of integration.

The unknown quantities Q_0, P_0, P_M, $\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}Q_n$ can be evaluated as before.

Case II Relation (38), on applying the theory of Laplace transforms gives

$$R(z)=\frac{\begin{array}{l}\mu_2(1-z)\{\alpha z+z(\lambda_1+\mu_1 p+\beta+\xi)-\lambda_1 z^2-\mu_1 p\}Q_0+\\ \mu_1 p(1-z)[\beta z-\{\mu_2-z(\mu_2+\alpha+\xi)\}]P_0+\lambda_1 z^{M+1}(1-z)\{\mu_2-z(\mu_2+\alpha+\xi)-\beta z\}P_M+\\ \xi z/(\alpha+\beta+\xi)[\beta\{\lambda_1 z^2-z(\lambda_1+\mu_1 p+\alpha+\beta+\xi)+\mu_1 p\}+(\alpha+\xi)\{\mu_2-z(\mu_2+\alpha+\beta+\xi)\}]\end{array}}{\begin{array}{l}z^3\lambda_1(\mu_2+\alpha+\xi)-z^2[\lambda_1(\mu_2+\alpha+\xi)+\{\alpha\mu_1 p+\mu_2(\lambda_1+\mu_1 p+\beta+\xi)\}+\xi(\alpha+\beta+\xi)]\\ +z[\{\alpha\mu_1 p+\mu_2(\lambda_1+\mu_1 p+\beta+\xi)\}+\mu_1 p\mu_2]-\mu_1 p\mu_2\end{array}} \quad \text{.... (43)}$$

or, we can write

$$R(z)=\frac{T(z)Q_0+N(z)P_0+L(z)P_M+M(z)}{K(z)} \quad \text{.... (44)}$$

Where T(z), N(z) and L(z) are the co-efficient of Q_0, P_0 and P_M respectively in the numerator of equation (43) and K(z) is the denominator of (43).

Equation (44) is a polynomial in z and exists for all values of z, including three zeros of the denominator. Hence Q_0, P_0 and P_M can be obtained by setting the numerator equal to zero.

Substituting the three zeros b_1, b_2 and b_3 (say) of the denominator (at each of which the numerator must vanish).

Three equations determining the constants Q_0, P_0 and P_M are

$$T(b_1)Q_0 + N(b_1)P_0 + L(b_1)P_M = -M(b_1) \quad .... (45)$$

$$T(b_2)Q_0 + N(b_2)P_0 + L(b_2)P_M = -M(b_2) \quad .... (46)$$

$$T(b_3)Q_0 + N(b_3)P_0 + L(b_3)P_M = -M(b_3) \quad .... (47)$$

After solving these equations, we have

$$Q_0 = \frac{-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}}{A}$$

$$P_0 = \frac{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}}{A}$$

$$P_M = \frac{-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}}{A}$$

where

$$A = \begin{vmatrix} T(b_1) & N(b_1) & L(b_1) \\ T(b_2) & N(b_2) & L(b_2) \\ T(b_3) & N(b_3) & L(b_3) \end{vmatrix}$$

A_{ij} is the co-factor of the $(i, j)^{th}$ element of A.

By putting the values of Q_0, P_0 and P_M in equation (44), we have

$$R(z) = \frac{\begin{array}{l} T(z)[-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}] + N(z)[M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}] \\ + L(z)[-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}] + A \cdot M(z) \end{array}}{A \cdot K(z)} \quad . ... (48)$$

6. Mean Queue Length:

Define,

L_q= Expected number of customers in the queue including the one in service.

Then

$$L_q = R'(z)\big|_{z=1}$$

Therefore, from equation (48), we have

$$L_q = \frac{\begin{array}{l} K(1)[T'(1)\{-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}\} + N'(1)\{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\} \\ + L'(1)\{-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}\} + A\,M'(1)] - [T(1)\{-M(b_1)A_{11} + M(b_2)A_{21} \\ - M(b_3)A_{31}\} + N(1)\{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\} + L(1)\{-M(b_1)A_{13} + M(b_2)A_{23} \\ - M(b_3)A_{33}\} + A \cdot M(1)]\,K'(1) \end{array}}{A \cdot [K(1)]^2}$$

.... (49)

where dashes denotes the first derivative w. r. t. z.

Relation (39), on applying the theory of Laplace transforms gives

$$r(z)=\frac{(1-z)\mu R_0-(1-z)\lambda_1 z^{M+1}P_M-\xi z}{\lambda_1 z^2-z(\lambda_1+\mu+\xi)+\mu} \qquad .... (50)$$

where

$$r(z)=\lim_{s\to 0} s\, r(z,s)$$

Equation (50) is a polynomial in z and exists for all values of z, including the two zeros of the denominator. Hence R_0 and P_M can be obtained by setting the numerator equal to zero. Two equations determining the constants R_0 and P_M are Substituting the two zeros a_1 and a_2 (say) of the denominator (at each of which the numerator must vanish).

$$(1-a_1)\mu R_0-(1-a_1)\lambda_1 a_1^{M+1}P_M=\xi a_1 \qquad .... (51)$$

$$(1-a_2)\mu R_0-(1-a_2)\lambda_1 a_2^{M+1}P_M=\xi a_2 \qquad .... (52)$$

On solving these equations, we have

$$P_M=\frac{(a_1-a_2)}{a_1^{M+1}-a_2^{M+1}} \text{ and } R_0=\frac{a_2\,\xi}{(1-a_2)\mu}+\frac{\lambda_1}{\mu}a_2^{M+1}\frac{a_1-a_2}{a_1^{M+1}-a_2^{M+1}} \;;$$

where

$$\lambda_1(1-a_1)\,(1-a_2)=-\xi,\quad a_1>1$$

Now, from equation (50), we have

$$r(z)=\frac{\xi+\lambda_1(1-z)(1-a_2)\dfrac{(a_1-a_2)}{a_1^{M+1}-a_2^{M+1}}\,\dfrac{a_2^{M+1}-z^{M+1}}{a_2-z}}{\lambda_1(z-a_1)\,(a_2-1)} \qquad .... (52)$$

r_n = The co-efficient of z^n

$$r_n=\frac{a_2}{\mu(1-a_2)}\left[\xi(1-P_M)+a_2^{M-n}\frac{a_1^{n+1}-a_2^{n+1}}{a_1-a_2}\right]\left(\frac{\lambda_1}{\mu}\right)^n a_2^n \qquad .... (54)$$

If $\xi=0$ and and p=1 (i.e., no catastrophe and feedback are allowed), then from equation (50), we have

$$r(z)=\frac{\mu R_0-\lambda_1 z^{M+1}P_M}{\mu-\lambda_1 z} \qquad .... (55)$$

The condition, $\lim_{z\to 1} r(z)=1$ gives

$$\mu R_0-\lambda_1 P_M=\mu-\lambda_1 \qquad .... (56)$$

As r(z) is analytic, the numerator and denominator of equation (55) must vanish simultaneously for $z=\mu/\lambda_1$, which is a zero of its denominator. Equating the numerator of equation (55) to zero for $z=\mu/\lambda_1$ we have

$$R_0=\rho^{-M}P_M, \qquad \rho=\lambda_1/\mu<1 \qquad .... (57)$$

Relation (56) and (57) gives

$$R_0 = \frac{1-\rho}{1-\rho^{M+1}} \quad , \quad P_M = \frac{(1-\rho)\rho^M}{1-\rho^{M+1}}$$

Now, from equation (55), we have

$$r(z) = \frac{1-\rho}{1-\rho^{M+1}} \; . \; \left[\frac{1-(\rho\, z)^{M+1}}{1-\rho z}\right] \qquad \text{.... (58)}$$

which is a well known result of the M/M/1 queue with finite waiting space M.

When there is an infinite waiting space, the corresponding expression for r(z) is obtained by letting M tends to infinity in equation (58), If $(\rho,|z|) < 1$.

$$r(z) = \frac{1-\rho}{1-\rho z} \qquad \text{.... (59)}$$

which is again a well known result of the M/M/1 queue with infinite waiting space.

Case III Relation (40), on applying the theory of Laplace Transform gives

$$R(z) = \frac{H_1(z)Q_0 + H_2(z)P_0 + H_3(z)P_M + H_4(z)\sum_{n=0}^{M} P_n + H_5(z)\sum_{n=0}^{M} Q_n + H'}{B_1(z)\; H(z)} \qquad \text{.... (60)}$$

where,

$$B_1(z) = B(z)\big|_{s=0}$$

$$H(z) = L''(z)\big|_{s=0}$$

$$H_i(z) = B(z)L''_{i+1}(z) + L''_{i+7}(z)\big|_{s=0} \qquad ; \text{ i=1, 2, 3, 4, 5.}$$

$$L''_j(z) = \int \left[\frac{z_j}{\eta_2(z)} L(z)\right]_{\varepsilon=\varepsilon'=1,\, N=1} dz \qquad ; \text{ j= 2, 3, 4, 5, 6.}$$

$$L''_k(z) = L_k(z)\big|_{\varepsilon=\varepsilon'=1,\, N=1} \qquad ; \text{ k= 8, 9, 10, 11, 12.}$$

H' = the constant of Integration.

The unknown quantities of equation (60) can be evaluated as before.

7. Conclusion and application of the model:

In this paper, we have studied the transient analysis of a limited capacity queueing system incorporating the effects of feedback, two randomly changing states and catastrophes. We have obtained some particular cases and steady state solutions along with some measures of effectiveness. The proposed model is highly useful for dealing with real-world queueing situations such as manufacturing systems with rework operations, shift changes and possibilities of feedback and catastrophes that occurs in production processes subject to rework, computer networks, telecommunication systems, banking sectors, hospital management, etc.

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