International Journal of
INTELLIGENT SYSTEMS AND APPLICATIONS IN ENGINEERING

ISSN:2147-6799 www.ijisae.org Original Research Paper

# Adjoint Maps on *H*-Semi Vector Spaces

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**Submitted:**05/09/2024 **Revised:**20/10/2024 **Accepted:**28/10/2024

**Abstract:** In practical situations we often encounter semi vector spaces. Semi vector spaces are algebraic structures analogous to vector spaces with the base fields replaced by semifields. Semi vector spaces with an inner product are called inner product semi vector spaces. Metrizable inner product semi vector spaces which are complete with respect to the induced metric are called *H*-semi vector spaces.

In this paper we discuss certain fundamental properties of adjoint maps on *H*-semi vector spaces.

***Keywords***: *vector, encounter, situations*

## 1.Introduction

A non-empty set $F$ with two binary operations $+$ and $\cdot$ defined on it is called a **semifield** if the following conditions are satisfied:

$(F, +)$ is a commutative semigroup.

$(F - \{0\}, \cdot)$ is a commutative group, where 0 is the identity element with respect to +, if it exists.

A **semi vector space** over a semifield $F$ is defined to be a non-empty set $X$ equipped with the operations $+ : X \times X \to X$, called addition and $\cdot : F \times X \to X$, called scalar multiplication, satisfying the following conditions:

For each $\alpha, \beta \in F, x, y, z \in X$,

$x + (y + z) = (x + y) + z;$ $x + y = y + x$

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$(\alpha\beta)\, x = \alpha\, (\beta x)$

$1x = x$, where 1 is the multiplicative identity of $F$ if exists.

$\alpha\,( x + y) = \alpha x + \alpha y;$ $(\alpha + \beta)x = \alpha x + \beta x.$

We shall write $\alpha\, x$ instead of $\alpha \cdot x$, for $x \in X$ and $\alpha \in F$.

Let $X$ be a semi vector space over the semifield $\mathbb{R}_+$. An inner product on $X$ is a function $\langle , ] : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

$\langle x, x ] \geq 0$ for all $x \in X$

$\langle x, x ] = 0$ if and only if $x = 0$

$\langle x, y ] = \langle y, x ]$ for all $x, y \in X$

$\langle \alpha x, y ] = \alpha \langle x, y ]$ for all $x, y \in X$ and $\alpha \in \mathbb{R}_+$

$\langle x + z, y ] = \langle x, y ] + \langle z, y ]$ for all $x, y, z \in X$ .

Hence $\mathbb{R}_+$ is the set of all non-negative real numbers.

A semi vector space with an inner product defined on it is called an inner product semi vector space.

Let $X$ be a semi vector space over the semifield $\mathbb{R}_+$. An $s$-norm on $X$ is a function $\|.|: X \to \mathbb{R}_+$ satisfying the following:

(i) $\|x| \geq 0$ for all $x \in X$ and $\|x| = 0$ if and only if $x = 0$

(ii) $\|\alpha x| = \alpha\|x|$ for all $x \in X$ and $\alpha \in \mathbb{R}_+$

(iii) $\|x + y| \leq \|x| + \|y|$ for all $x, y \in X$.

Let $(X, \langle\ ,])$ be an inner product semi vector space. $X$ is said to be metrizable if there exists a metric function $d: X \times X \to \mathbb{R}$ such that

i) $\left(d(x,y)\right)^2 = 2\left(\|x|^2 + \|y|^2\right) - \|x + y|^2$ for all $x, y \in X$

and ii) $|\langle x, y] - \langle x, z]| \leq \|x|\ d\ (y, z)$ for all $x, y, z \in X$, where $\|x| = \sqrt{\langle x, x]}$.

In this case we say that $d$ is the induced metric of $\langle, ]$.

A Hilbert semi vector space is an inner product semi vector space $X$, which is complete with respect to the induced metric.

Let us call a Hilbert semi vector space a $H$-semi vector space.

## 2.Adjoints of linear Maps

### 2.1 Definition

A linear map between two semi vector spaces $X$ and $Y$ over $\mathbb{R}_+$ is a map $F: X \to Y$ satisfying $F(x_1 + x_2) = F(x_1) + F(x_2)$ for all $x_1, x_2 \in X$ and $F(\alpha x) = \alpha\, F(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}_+$.

### 2.2 Definition

Let $A$ be a bounded linear map on a $H$-semi space $X$ over $\mathbb{R}_+$. Suppose there exists a bounded linear map $B$ on $X$ such that

$$\langle Ax, y] = \langle x, By] \text{ for all } x, y \in X. \quad (1)$$

Then, $B$ is called an adjoint map of $A$.

It is to be noted that the existence of an adjoint map is not guaranteed.

### 2.3 Remark

Since $X$ is a semi vector space over $\mathbb{R}_+$, $\langle x, y] = \langle y, x]$ for all $x, y \in X$. So (1) becomes,

$$\langle Ax, y] = \langle x, By] = \langle By, x] \text{ for all } x, y \in X. \quad (2)$$

### 2.4 Definition

An inner product semi vector space $(X, \langle, ])$ is standard if $\langle x, z] = \langle y, z]$ for all $z \in X \Longrightarrow x = y$. A $H$-semi vector space which is standard is called a standard $H$-semi vector space or $H$-semi space.

### 2.5 Proposition

Let $X$ be a standard $H$-semi space. Then, the adjoint of any bounded linear map, if it exists, is unique.

**Proof**

Let $A$ be a bounded linear map on $X$. Suppose $B$ and $C$ are two adjoint maps of $A$.

Fix $y$. Then, $\langle Ax, y] = \langle x, By] = \langle x, Cy]$ for all $x \in X$.

Since $x \in X$ is arbitrary, and $X$ is standard, we get, $By = Cy$.

Hence, since $y \in X$ is arbitrary, $B = C$.

□

**Notation**

Let us denote the adjoint map of $A$, if it exists, by $A^*$. Then (2) becomes,

$$\langle x, Ay] = \langle A^*x, y] \text{ for all } x, y \in X. \quad (3)$$

**2.6 Theorem**

Let $X$ be a standard $H$-semi space over $\mathbb{R}_+$. Let $A$ and $B$ be two bounded linear maps on $X$ with adjoint operators $A^*$ and $B^*$ respectively. Then,

i) $A + B$ also has an adjoint and $(A + B)^* = A^* + B^*$

ii) $\alpha A$ has an adjoint and $(\alpha A)^* = \alpha A^*$ for $\alpha \in \mathbb{R}_+$

iii) $(A^*)^* = A$

iv) $AB$ has an adjoint and $(AB)^* = B^*A^*$.

**Proof**

The proof is similar to that in vector spaces.

As $A$ and $B$ have adjoints $A^*$ and $B^*$ respectively,

we have $\langle Ax, y ] = \langle x, A^*y ]$ and $\langle Bx, y ] = \langle x, B^*y ]$ for all $x, y \in X$.

i) Since $A^*$ *and* $B^*$ are bounded maps on $X$, $A^* + B^*$ is also a bounded map on $X$.

Now, for $x, y \in X$, consider

$\langle (A + B)x, y ] = \langle Ax, y ] + \langle Bx, y ] = \langle x, A^*y ] + \langle x, B^*y ]$

$= \langle x, A^*y + \langle B^*y ] = \langle x, (A^* + B^*)y ]$. (4)

Hence , $(A + B)^* = A^* + B^*$.

ii) Let $\alpha \in \mathbb{R}_+$.

Then for $x, y \in X$,

$\langle(\alpha A)x, y ] = \alpha \langle Ax, y ] = \alpha\langle x, A^*y ]$

$= \langle x, \alpha A^*y ]$.

So, $(\alpha A)^* = \alpha A^*$.

iii) Consider

$\langle A^*x, y ] = \langle y, A^*x ] = \langle Ay, x ] = \langle x, Ay ]$ for all $x, y \in X$.

Hence A is the adjoint operator of $A^*$. That is, $(A^*)^* = A$.

iv) AB and $B^*A^*$ are bounded maps on $X$.

Now consider, for $x, y \in X$,

$\langle x, (AB)y ] = \langle x, A(By)] = \langle A^*x, By ] = \langle B^*A^*x, y ]$.

Hence , $(AB)^* = B^*A^*$. ☐

**2.7 Theorem**

Let $A$ be a bounded linear map on a standard $H$-semi vector space $X$ over $\mathbb{R}_+$ with adjoint $A^*$. Then,

(i) $\|A^*| = \|A|$

(ii) $\|A^*A| = \|Ax|^2$

(iii) $A^*A = 0$ *if and only if* $A = 0$.

**Proof**

For $x \in X$, Consider $\|Ax|^2 = \langle Ax, Ax ] = \langle A^*Ax, x ]$

$\leq \|A^*Ax|\|x|$, using Schwarz inequality

$\leq \|A^*A|\|x|^2$.

So, $\|Ax| \leq \sqrt{\|A^*A|}\, \|x|$ for all $x \in X$.

Hence $\|A| \leq \sqrt{\|A^*A|}$ . So, $\|A^2| \leq \|A^*A|$.

(5)

(5) implies,

$\|A|^2 \leq \|A^*A| \leq \|A^*|\|A|$. Hence, $\|A| \leq \|A^*|$ for $A \neq 0$. (6)

For $A = 0$, (6) is obvious.

Hence $\|A| \leq \|A^*|$ (7)

Replacing $A$ by $A^*$, we get,

$\|A^*| \leq \|(A^*)^*| = \|A|$, since $A^{**} = A$. (8)

From (7) and (8),

$$\|A^*| = \|A|.$$

Now (5) implies,

$$\|A|^2 = \|A^*A| \leq \|A^*|\|A| = \|A|^2 \text{ since } \|A^*| = \|A|.$$

Hence all the terms in the above chain are equal.

So $\|A^*A| = \|A|^2$.

Again, $A^*A = 0$ *if and only if* $\|A^*A| = 0$,

*if and only if* $\|A|^2 = 0$

*if and only if* $A = 0$.

☐

**2.8 Example**

$0^* = 0$ and $I^* = I$, where I is the identity operator.

**3.Convergence of Linear Maps**

**3.1 Definition**

For $A$ and $B \in B(X)$, $D(A, B)$ is defined by $D(A, B) = sup\{d(A(x), B(x))/\, x \in X, \|x| \leq 1\}$, where $\|x| = \sqrt{\langle\, x, x\,]}$.

Then $D$ is a metric on $B(X)$. Here $d(x, y) = 2\|x|^2 + 2\|y|^2 - \|x + y|^2$ (9)

**3.2 Proposition**

Let $A, B \in B^*(X)$.

Then $D(A, B) = D(A^*, B^*)$

**Notation**

Let $X$ be a $H$-semi space over $\mathbb{R}_+$. The set of all bounded maps $A: X \to X$ is denoted by $B(X)$. Let us denote the set of all bounded maps $A: X \to X$ for which $A^*$exists , by $B^*(X)$.

**3.3 Remark**

(a) 0 and $I \in B^*(X)$. Thus, $B^*(X)$ is non empty.

(b) $B^*(X)$ is a subspace of $B(X)$.

**3.4 Definition**

$X$ is said to have the weak convergence property if ($\langle\, x, y_n\,]$) convergent for all $x \in X$ implies $(y_n)$ is convergent.

**3.5 Definition**

$X$ is said to have the convergence property if $\langle\, x, y_n\,] \to \langle\, x, y\,]$ *for all* $x$ implies $y_n \to y$.

**3.6 Theorem**

Suppose $X$ has the weak convergence property. Then, $B^*(X)$ is a closed subspace of $B(X)$.

**Proof**

Let $(T_n)$ be a sequence in $B^*(X)$ such that $T_n \to T$ in $B(X)$.

Consider, for $x, y \in X$,

$$\langle\, Tx, y\,] = \left\langle \lim_{n\to\infty} T_n x, y\,\right] = \lim_{n\to\infty} \langle\, T_n x, y\,]$$

$$= \lim_{n\to\infty} \langle\, x, T_n^* y\,]. \qquad (10)$$

So $\lim_{n\to\infty} \langle\, x, T_n^* y\,]$ exists for all $x \in X$.

As $X$ has the weak convergence property, $(T_n^* y)$ is convergent.

Define $Sy = \lim_{n\to\infty} T_n^* y$ for all $y \in X$ . Then S is linear. (11)

Since $(T_n)$ is convergent, $(T_n)$ is bounded.

So, there exists $c < \infty$ such that

$\|T_n| < +c\ \forall n.$ (12)

Now, $\|T_n^*| = \|T_n| \leq c \; \forall n.$

(13)

From (11) we get,

$\|Sy| = \left\|\lim_{n\to\infty} T_n^* y\right| = \lim_{n\to\infty} \|T_n^* y|$

$\leq \lim_{n\to\infty} \|T_n^*| \|y| \leq c\|y|$ for all $y \in X.u$sing (13).

Hence S is bounded.

So (10) becomes,

$\langle Tx, y ] = \langle x, Sy ].$ Hence $S = T^*$, since $y \in X$ is arbitrary.

Thus, $T^* y = \lim_{n\to\infty} T_n^* y$ for all $y \in X$.

Now $D(T_n^*, T^*) = D(T_n, T) \to 0$ as $n \to \infty$.

Hence $(T_n^*) \to T^*$ and hence $B^*(X)$ is closed in $B(X)$.

□

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