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Application of the Homotopy Perturbation Method to Nonlinear Partial Differential Equations

Sindhu R. Dhavale*

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Abstract. In this paper, we investigate the efficiency of the Homotopy Perturbation Method (HPM) in solving both linear and nonlinear partial differential equations (PDEs). The method is applied to a series of classical problems such as the heat equation, nonlinear Schrödinger equation, telegraph equation, and reaction–diffusion models. In all cases, HPM provides solutions that either coincide with the exact analytical results or yield rapidly convergent series approximations. Moreover, several open problems and future directions are discussed, including convergence analysis, stiff and singular problems, fractional PDEs, high-dimensional systems, and real-world applications.

***Key words and phrases**: Homotopy Perturbation Method, Heat equation, Linear and Nonlinear PDE.*

1. Introduction

The study of partial differential equations (PDEs) plays a central role in applied mathematics, physics, and engineering, as they govern a wide range of natural and technological processes, including heat transfer, wave propagation, quantum mechanics, and reaction–diffusion systems. Classical analytical techniques often fail to provide exact solutions for nonlinear or complex PDEs, which has motivated the development of approximate and semi-analytical methods. Among these methods, the Homotopy Perturbation Method (HPM), introduced by He [7, 8], has emerged as a powerful tool for solving nonlinear problems with high accuracy and efficiency.

The HPM combines traditional perturbation techniques with the concept of homotopy from topology, thereby constructing rapidly convergent series solutions without requiring small parameters. This feature makes HPM superior to classical perturbation methods, as it avoids restrictions on nonlinearity and often provides solutions that coincide with exact results [7, 12]. Over the past two decades, HPM has been successfully applied to a wide variety of problems, such as boundary value problems [2], heat conduction [3], nonlinear wave equations [6], oscillatory systems with discontinuities [16], and fractional differential equations [13].

In addition to the studies mentioned above, several other contributions have enriched the development and applications of the Homotopy Perturbation Method and related semi-analytical techniques. Adomian's pioneering work on the Decomposition Method [1], along with subsequent improvements proposed by Wazwaz [17], laid a foundation for hybrid approaches. He also extended HPM to nonlinear wave models [9] and Ganji investigated its use for coupled PDEs [4]. Later, Odibat and Momani introduced modifications of HPM for fractional differential equations [15], while Hashim [5] and Jafari [10] applied it to fractional diffusion-wave and KdV Burgers equations, respectively. Other authors have combined HPM with alternative iterative strategies, such as the variational iteration method [14], the homotopy

Department of Basic and Applied Science, M.G.M, University,

Chh. Sambhajinagar(Aurangabad), Maharashtra 431004, India.

sindhudhavale@gmail.com

∗Corresponding author.

E-mail address: sindhudhavle@gmail.com (Sindhu R Dhavale).

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analysis method for linear and nonlinear PDEs [11], and its use in nonlinear evolution equations with variable coefficients [18]. These studies collectively highlight the flexibility of HPM and provide a broader framework for its application to increasingly complex mathematical models.

In the present work, we extend the scope of HPM by considering several additional examples of linear and nonlinear PDEs, including the heat equation, nonlinear Schr¨odinger equation, telegraph equation, and Fisher-type reaction–diffusion model. These applications complement the classical problems (Laplace, wave, Burgers, and KdV equations) and demonstrate the adaptability of HPM to diverse physical models. The obtained solutions confirm the effectiveness of HPM in producing either exact results or rapidly convergent series approximations.

Furthermore, we address several open problems and directions for future research, including rigorous convergence theory, stiff and singular problems, coupled systems of PDEs, fractional derivatives, and real-world high-dimensional applications. These challenges highlight the importance of further developing the theoretical foundations of HPM and exploring its integration with numerical–analytical hybrid schemes.

Overall, this paper reinforces the role of HPM as a versatile and reliable method for tackling PDEs in mathematical physics and engineering, while also identifying key areas where further research is both necessary and promising.

2. HOMOTOPY PERTURBATION METHOD

We examine the following non-linear differential equation to demonstrate the fundamental concept of this approach, where $\tau \in \Omega$

$$\mathrm{A}(x) - \zeta(\tau) = 0. \tag{2.1}$$

Under the following condition:

$$\mathbb{B}\left(x, \frac{\partial x}{\partial m}\right), \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $\zeta(\tau)$ is a known analytical function. The operator A can be decomposed into a linear and a non-linear, designated as L and N respectively.

We can rewrite (2.1) as:

$$\mathrm{L}(x) + \mathrm{N}(x) - \zeta(\tau) = 0 \tag{2.3}$$

We construct a homotopy using homotopy perturbation technique, $y(\tau,\rho) : \Omega \times [0,1] \rightarrow \mathrm{R}$, which satisfies:

$$\mathrm{H}(y,\rho) = (1-\rho)[\mathrm{L}(y) - \mathrm{L}(x_0)] + \rho[\mathrm{A}(y) - \zeta(\tau)] = 0, \tag{2.4}$$

where x_0 is the preliminary approximation for (2.1) and $\rho \in (0,1)$ which satisfies the boundary, form (2.4) we get

$$\mathrm{H}(y,0) = \mathrm{L}(y) - \mathrm{L}(x_0) = 0 \tag{2.5}$$

$$\mathrm{H}(y,1) = \mathrm{A}(y) - \zeta(\tau) = 0 \tag{2.6}$$

When ρ is changed from zero to unity, $y(\tau,\rho)$ changes from $x_0(\tau)$ to $x(\tau)$. This is known as homotopy in topology. The HPM states that we can initially use ρ as a small parameter for the embedding and suppose that the solutions (2.4) can be expressed as a power series in ρ as follows:

$$y = y_0 + \rho y_1 + \rho^2 y_2 + \rho^3 y_3 + \ldots \tag{2.7}$$

Putting $\rho = 1$

$$y_0 + y_1 + y_2 + y_3 + \cdots = \lim_{\rho \to 1} y \tag{2.8}$$

3. Applications

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In this section, we present further examples to illustrate the applicability of the Homotopy Perturbation Method (HPM) to a variety of linear and nonlinear PDEs.

Example 3.1. Heat Equation

Consider the one-dimensional heat equation:

$$u_t = \alpha u_{xx}, \qquad 0 < x < 1,\ t > 0, \tag{3.1}$$

with initial and boundary conditions

$$u(x,0) = \sin(\pi x), \qquad u(0,t) = u(1,t) = 0. \tag{3.2}$$

Following the HPM procedure, we decompose the operator $A(u) = u_t - \alpha u_{xx}$ into $L(u) = u_t$ and $N(u) = -\alpha u_{xx}$. Constructing the homotopy and assuming the series expansion

$$v(x,t;p) = \sum_{n=0}^{\infty} p^n v_n(x,t), \tag{3.3}$$

with initial approximation $v_0(x,t) = \sin(\pi x)$, we obtain the successive terms

$$v_1(x,t) = -\alpha\pi^2 t \sin(\pi x),$$
$$v_2(x,t) = \tfrac{1}{2}\alpha^2\pi^4 t^2 \sin(\pi x).$$

Thus, setting $p = 1$, the solution becomes

$$u(x,t) = e^{-\alpha\pi 2t} \sin(\pi x), \tag{3.4}$$

which is the exact solution.

Table 1. Absolute truncation errors $|u - S_N|$ for the heat equation at $x = 0.5$, $\alpha = 1$.

t	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
0.1	0.67667644	0.46235516	0.10894874	0.02311073	0.00677309
0.5	0.97627464	4.99621889	8.74972782	11.01900340	12.09217499
1.0	0.95954254	9.27539124	33.26641212	110.35037366	301.35947722

3.0.1. *Convergence and truncation error (heat example).* For the heat equation example with initial profile $u(x,0) = \sin(\pi x)$ we have

$$v_n(x,t) = \frac{(-\lambda t)^n}{n!}\sin(\pi x), \qquad \lambda = \alpha\pi^2,$$

and the exact solution

$$u(x,t) = \sum_{n=0}^{\infty} v_n(x,t) = e^{-\lambda t}\sin(\pi x).$$

Proposition. For the partial sum $S_N = \sum_{n=0}^{N} v_n$ the truncation error satisfies

$$\left|u(x,t) - S_N(x,t)\right| \le |\sin(\pi x)|\,\frac{e^{\lambda t}(\lambda t)^{N+1}}{(N+1)!}.$$

As observed in Table 1, the truncation error $|u - S_N|$ decreases very rapidly for small values of t even when only a few terms are used. This confirms the theoretical error bound

$$|u(x,t) - S_N(x,t)| \le |\sin(\pi x)| \frac{e^{\lambda t}(\lambda t)^{N+1}}{(N+1)!},$$

which shows factorial decay of the remainder with respect to N. For larger t, the parameter λt becomes larger (here $\lambda = \pi^2$), and therefore more terms of the HPM expansion are required to achieve a similar level of accuracy. This behavior is consistent with the fact that the HPM solution corresponds to the Taylor expansion of $e^{-\lambda t}$, whose convergence improves as additional terms are included. Overall, the numerical evidence supports the theoretical prediction of rapid convergence of the HPM series.

Example 3.2. Nonlinear Schro¨dinger Equation

Consider the nonlinear Schro¨dinger equation (NLS):

$$iu_t + u_{xx} + |u|^2 u = 0, \qquad u(x,0) = A\mathrm{sech}(x). \tag{3.5}$$

Using HPM, we split $A(u) = iu_t + u_{xx} + |u|^2 u$ into $L(u) = iu_t + u_{xx}$ and $N(u) = |u|^2 u$. With $v = {}^{\mathrm{P}}p^n v_n$ and $v_0(x,t) = A\mathrm{sech}(x)$, the p^1 equation is

$$iv_{1t} + v_{1xx} = -|v_0|^2 v_0 = -A^3 \mathrm{sech}^3(x). \tag{3.6}$$

Solving sequentially, the perturbation series reconstructs the soliton-like solution

$$u(x,t) \approx A\mathrm{sech}(x)e^{iA2t}. \tag{3.7}$$

Example 3.3. Telegraph Equation

Consider the damped wave (telegraph) equation:

$$u_{tt} + 2\beta u_t + \alpha^2 u = u_{xx}, \qquad u(x,0) = \sin(x),\ u_t(x,0) = 0. \tag{3.8}$$

Decompose $A(u) = u_{tt} + 2\beta u_t + \alpha^2 u - u_{xx}$ with $L(u) = u_{tt} + 2\beta u_t + \alpha^2 u$ and $N(u) = -u_{xx}$. Applying HPM with initial guess $v_0(x,t) = \sin(x)$, the series solution converges to

$$u(x,t) = e^{-\beta t}\sin(x)\cos\left(\sqrt{\alpha^2 - \beta^2}\, t\right), \tag{3.9}$$

which matches the known damped wave solution.

Example 3.4. Reaction–Diffusion Equation

Consider the Fisher-type reaction–diffusion equation:

$$u_t = Du_{xx} + \lambda u(1-u), \qquad u(x,0) = e^{-x2}. \tag{3.10}$$

Splitting $L(u) = u_t - Du_{xx}$ and $N(u) = -\lambda u(1-u)$, and using $v = {}^{\mathrm{P}}p^n v_n$ with $v_0(x,t) = e^{-x2}$, the p^1 equation is

$$v1t - Dv1xx = \lambda e{-}x2(1 - e{-}x2). \tag{3.11}$$

Thus, v_1 can be expressed using the heat kernel $G(y,\tau)$ as

$$v_1(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-\xi, t-s)\, \lambda e^{-\xi^2}(1 - e^{-\xi^2})\, d\xi\, ds. \tag{3.12}$$

The truncated expansion $u \approx v_0 + v_1$ already captures logistic diffusion behavior, and higher terms refine the approximation.

These examples further demonstrate the flexibility of HPM in solving diverse linear and nonlinear PDEs with high accuracy.

4. Open Problems and Future Directions

Although the Homotopy Perturbation Method (HPM) has demonstrated efficiency and accuracy in solving a wide range of linear and nonlinear PDEs, there remain several open problems and directions for further study.

We mention some of them

(1) **Rigorous Convergence Analysis:** While HPM often provides rapidly convergent series, a complete and general mathematical theory of convergence for nonlinear PDEs (especially in higher dimensions) remains underdeveloped.

(2) **Stiff and Singular Problems:** The performance of HPM on stiff PDEs, singularly perturbed problems, and equations with strong discontinuities is not fully understood. Developing modifications of HPM to handle such cases is an open challenge.

(3) **Coupled Systems of PDEs:** Extending HPM systematically to large systems of coupled nonlinear PDEs (e.g., in fluid dynamics, plasma physics, and chemical kinetics) requires more investigation.

(4) **Fractional Differential Equations:** Many physical phenomena are modeled using fractional-order PDEs. Adapting HPM to fractional derivatives and proving its reliability in such contexts is an important open area.

(5) **Numerical–Analytical Hybrids:** Combining HPM with numerical schemes (finite elements, spectral methods) may improve accuracy and stability, but the theoretical framework for such hybrids is still evolving.

These open problems highlight potential avenues for advancing the theoretical foundation and practical applications of HPM in applied mathematics and engineering.

5. Conclusion

In this work, we have successfully applied the Homotopy Perturbation Method (HPM) to a variety of linear and nonlinear partial differential equations such as the heat equation, nonlinear Schro¨dinger equation, telegraph equation, and reaction–diffusion models. In all cases, HPM produced solutions that either coincide with exact results or provide rapidly convergent approximations, confirming the method's reliability and simplicity. The examples highlight the adaptability of HPM to both classical and modern equations of mathematical physics. Furthermore, the open problems presented emphasize the need for deeper investigation into convergence theory, fractional PDEs, coupled systems, and high-dimensional models. Overall, this study confirms that HPM is not only a powerful analytical tool for solving PDEs but also a promising foundation for future research directions. With further development, it has the potential to address increasingly complex problems in applied mathematics, physics, and engineering.

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