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Differential Neighborhoods and Differential Limit Points in Derivative Topological Spaces

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Abstract: In recent years, some generalized structures of topologies were introduced. In this way, derivative topological space was introduced. To contribute in this orientation, we introduce and investigate the properties of differential neighborhoods and differential limit points in derivative topological spaces. We explore many properties of them and discuss their behaviour on derivative topological spaces. A differential limit point is a point in a differential ring that can be approached arbitrarily close by the elements that same set. These concepts serve as the building blocks for defining other key properties in derivative topological spaces. The relation between differential open sets, differential closure, differential neighborhoods and differential derived sets were obtained.

Key words: *Derivative topology, differential open, differential neighborhood, differential limit point and differential derived set.*

1. Introduction

Throughout this paper, we focus only on differential rings. Let R denote the differential ring unless otherwise specified. The notion of the ring with derivation is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. In 1950's a new part of algebra called differential algebra was initiated by the works of Ritt and Kolchin. For many years, various authors [1,2,5] constructed some topologies over algebraic structures and they investigated the relations between the algebraic properties of given algebraic structures (such as rings, modules, lattices and fuzzy structures) and topological properties of these topologies.

In this manner, we introduce a new topology, namely, derivative topology τ which has the structure that $\emptyset$, R are in τ and is closed under arbitrary union and arbitrary intersection. We observe that a derivative topology is different from general topology.

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During 20th century, real and complex analysis relied heavily on the concepts of open sets, closed set, neighborhood and limit point of a set. The concept of limit point is taken as primitive for topological spaces. In this paper, we introduce the notion of differential neighborhood, differential limit point and study its properties extensively.

Section 2 deals with the preliminary concepts. In section 3, we introduce differential neighborhoods and study their basic properties. In section 4, we see the notions of differential limit point and discuss its properties.

2. Preliminaries

The purpose of this section is to give a review of some definitions and propositions concerning our subject. Throughout the work, R denote the differential ring.

Definition 2.1 [7]: Let R be a commutative ring with identity. A derivation on R is a map d: R→ R such that

(i) d(a+b)= d(a)+d(b)
(ii) d(a.b)= d(a).b+a.d(b)

A differential ring is a commutative ring with identity R together with the distinguished derivation d. If R is a differential ring and x ∈R, then write x' for dx, x'' for d^2x and in general $x^{(n)}$for d^nx.

Let R be a commutative ring with identity. A derivation d on R is said to be the trivial derivation if $d(a) = 0$ for all $a \in R$.

The derivative of a non-empty set S is denoted by d(S) and defined as d(S) = {d(a)/a$\in S$}.

Definition 2.2 [7]: A subset S of a differential ring R is called a differential subset if it contains the derivative of each of its elements. Equivalently, d(S)$\subseteq$ S, where d(S) is the derivative of S.

Proposition 2.3[10]: Let R be a differential ring with derivation d. Let τ be a collection of differential subsets of R. Then

1. $\emptyset, R \in \tau$
2. Arbitrary union of differential subsets of R is a differential subset of R.
3. Arbitrary intersection of differential subsets of R is a differential subset of R.

Definition 2.4 [10]: A derivative topology on a differential ring R is a collection τ of differential subsets of R having the following properties:

1. $\emptyset, R$ are in τ.
2. The union of the elements of any subcollection of τ is in τ.
3. The intersection of the elements of any subcollection of τ is in τ.

If R is a derivative topological space with derivative topology τ, we say that a differential subset U of R is a differential open set of R if U belongs to τ.

Definition 2.5 [10]: Let (R, d, τ) be a derivative topological space. A subset A of R is said to be differential closed if its complement is differential open of R.

Proposition 2.6 [10]: Let (R, d, τ) be a derivative topological space. Then the following properties hold:

1. $\emptyset$ and R are differential closed sets
2. Arbitrary intersection of differential closed sets is a differential closed set.
3. Arbitrary union of differential closed sets is a differential closed set.

Proposition 2.7 [11]: Let A be a subset of the derivative topological space R. Then $x \in Cl_d(A)$ iff every differential open set U containing x intersects A.

Proposition 2.8 [11]: Let X be a subset of a derivative topological space (R, d, τ). Then

i. $Cl_d(X) \supseteq X$.
ii. $Cl_d(X) = X$ if and only if X is a differential closed subset of R.

3. Differential Neighborhoods

In this section, we will define differential neighborhood and see some results which connect differential neighborhoods with differential open sets and differential closure.

Definition 3.1: Let (R, d, τ) be a derivative topological space and $x \in R$. A subset N of R is called a differential neighborhood of x if there exists a differential open set U such that $x \in U \subseteq N$.

Clearly, every differential neighborhood of x contains x.

Proposition 3.2: A subset N of a derivative topological space (R, d, τ) is differential open in R iff N is a differential neighborhood of each of its points.

Proof: Let N be a differential open set in R. Let $x \in N$. Obviously, $x \in N \subseteq N$. Therefore, N is a differential neighborhood of x. Hence, N is a differential neighborhood of each of its points. Conversely, suppose N is a differential neighborhood of each of its points. Then for any $x \in N$, there exists a differential open set U_x such that $x \in U_x \subseteq N$. Therefore, $\bigcup_{x\in N} U_x = N$. By Proposition 2.3 arbitrary union of differential open sets is differential open. Therefore, N is a differential open set.

Proposition 3.3: If A is a differential closed subset of a derivative topological space (R, d, τ) and $x \in R - A$, then there exists a differential neighborhood N of x such that $N \cap A = \emptyset$.

Proof: Let A be a differential closed subset of R. Then $R - A$ is a differential open set. Since $x \in R - A$, by Proposition 3.2, $R - A$ is a differential neighborhood of x. Take $N = (R - A)$. Then N is a differential neighborhood of x such that $N \cap A = \emptyset$.

Proposition 3.4: Let A be a subset of a derivative topological space (R, d, τ). Then $x \in Cl_d(A)$ iff every differential neighborhood of x intersects A.

Proof: By Proposition 2.7, $x \in Cl_d(A)$ iff every differential open set U containing x intersects A. Also, by Proposition 3.2, N is differential open in R iff N is a differential neighborhood of each of its points. Hence, $x \in Cl_d(A)$ iff every differential neighborhood of x intersects A.

Proposition 3.5: Let (R, d, τ) be a derivative topological space. Let $x \in R$. Then every point x has at least one differential neighborhood.

Proof: Since R is a differential open set, then by Proposition 3.2, it is a differential neighborhood of each of its points. Therefore, there exists at least one differential neighborhood namely, R for each $x \in R$.

Proposition 3.6: Let (R, d, τ) be a derivative topological space. Let $x \in R$. Then every super set of a differential neighborhood of x is a differential neighborhood of x.

Proof: Let N be a differential neighborhood of x and M contains N. Therefore, there exists a differential open set U such that $x \in U \subseteq N$. Since $N \subseteq M$, we have $x \in U \subseteq M$. Hence, M is a differential neighborhood of x.

Proposition 3.7: Let (R, d, τ) be a derivative topological space. Let $x \in R$. Then the intersection of two differential neighborhoods of x is also a differential neighborhood of x.

Proof: Let N and M be two differential neighborhoods of x. Then there exist differential open sets U and V in R such that $x \in U \subseteq N$ and $x \in V \subseteq M$. Therefore, $x \in U \cap V \subseteq N \cap M$. Since U and V are differential open sets in R, by Proposition 2.3 we have $U \cap V$ is differential open in R. Therefore, $N \cap M$ is a differential neighborhood of x.

Proposition 3.8: Let (R, d, τ) be a derivative topological space. Let $x \in R$. Then the union of two differential neighborhoods of x is also a differential neighborhood of x.

Proof: Let N and M be two differential neighborhoods of x. Then there exist differential open sets U and V in R such that $x \in U \subseteq N$ and $x \in V \subseteq M$. Therefore, $x \in U \cup V \subseteq N \cup M$. Since U and V are differential open sets in R, by Proposition 2.3 we have $U \cup V$ is differential open in R. Therefore, $N \cup M$ is a differential neighborhood of x.

Proposition 3.9: Let (R, d, τ) be a derivative topological space. Let $x \in R$ and let $N_d(x)$ be the set of all differential neighborhood of x. If $N \in N_d(x)$, then there exists $U \in N_d(x)$ such that $U \subseteq N$ and $U \in N_d(y)$ for every $y \in U$.

Proof: Let $N \in N_d(x)$. Then there exists differential open set U in R such that $x \in U \subseteq N$. Since U is a differential open set, by Proposition 3.2 it is a differential neighborhood of each of its points. Hence $U \in N_d(y)$ for every $y \in U$.

4. Differential limit Points

In this section we define differential limit points and differential derived set and establish its properties.

Definition 4.1: Let A be a subset of a derivative topological space (R, d, τ). Then x is a differential limit point of A if every differential neighborhood of x intersects A in some point other than x itself.

The set of all differential limit points of the set A is called a differential derived set of A and is denoted by $D_d(A)$.

Proposition 4.2: Let A be a subset of a derivative topological space (R, d, τ). Then $Cl_d(\mathrm{A}) = \mathrm{A} \cup \mathrm{D_d}(\mathrm{A})$.

Proof: Let $x \in \mathrm{A} \cup \mathrm{D_d}(\mathrm{A})$. If $x \in A$, then by Proposition 2.8, $x \in Cl_d(\mathrm{A})$. If $x \notin A$, then $x \in \mathrm{D_d}(\mathrm{A})$. Therefore, every differential neighborhood of x intersects A other than x. Hence by Proposition 2.7, $x \in Cl_d(\mathrm{A})$. Thus, $\mathrm{D_d}(\mathrm{A}) \subseteq Cl_d(\mathrm{A})$. Hence, $\mathrm{A} \cup \mathrm{D_d}(\mathrm{A}) \subseteq Cl_d(\mathrm{A})$. On the other hand, let $x \in Cl_d(\mathrm{A})$. We show that $x \in \mathrm{A} \cup \mathrm{D_d}(\mathrm{A})$. If $x \in A$, then $x \in \mathrm{A} \cup \mathrm{D_d}(\mathrm{A})$. If $x \notin A$, then we need to prove that $x \in \mathrm{D_d}(\mathrm{A})$. Since $x \in Cl_d(\mathrm{A})$, by Proposition 2.7 every differential neighborhood U of x intersects A. Since $x \notin A$, the differential neighborhood U of x must intersect A in some point other than x. Therefore, x is a differential limit point of A. Hence, $x \in \mathrm{D_d}(\mathrm{A})$. Therefore, $x \in \mathrm{A} \cup \mathrm{D_d}(\mathrm{A})$. Hence, $Cl_d(\mathrm{A}) \subseteq \mathrm{A} \cup \mathrm{D_d}(\mathrm{A})$. Thus, $Cl_d(\mathrm{A}) = \mathrm{A} \cup \mathrm{D_d}(\mathrm{A})$.

Proposition 4.3: A subset A of a derivative topological space (R, d, τ) is differential closed iff it contains all of its differential limit points.

Proof: A is differential closed iff $cl_d(A) = A$ iff $\mathrm{A} \cup \mathrm{D_d}(\mathrm{A}) = \mathrm{A}$ iff $\mathrm{D_d}(\mathrm{A}) \subseteq \mathrm{A}$.

Proposition 4.4: Let (R, d, τ) be a derivative topological space. Let A and B be subsets of R.

i. If $A \subseteq B$, then $\mathrm{D_d}(\mathrm{A}) \subseteq \mathrm{D_d}(\mathrm{B})$

ii. $\mathrm{D_d}(\mathrm{A} \cup \mathrm{B}) = \mathrm{D_d}(\mathrm{A}) \cup \mathrm{D_d}(\mathrm{B})$

iii. $\mathrm{D_d}(\mathrm{A} \cap \mathrm{B}) \subseteq \mathrm{D_d}(\mathrm{A}) \cap \mathrm{D_d}(\mathrm{B})$.

Proof: i. Let A and B be subsets of R such that $A \subseteq B$. Let $x \in \mathrm{D_d}(\mathrm{A})$. Then x is the differential limit point of A. Therefore, every differential neighborhood of x intersects A other than x. Since $A \subseteq B$, we have every differential neighborhood of x intersects B other than x. Therefore, x is a differential limit point of B. Hence, $x \in \mathrm{D_d}(\mathrm{B})$. Thus, $\mathrm{D_d}(\mathrm{A}) \subseteq \mathrm{D_d}(\mathrm{B})$.

ii. Since $A \cup B \supseteq A$ and $A \cup B \supseteq B$, by i, $\mathrm{D_d}(\mathrm{A} \cup \mathrm{B}) \supseteq \mathrm{D_d}(\mathrm{A})$ and $\mathrm{D_d}(\mathrm{A} \cup \mathrm{B}) \supseteq \mathrm{D_d}(\mathrm{B})$. Therefore, $\mathrm{D_d}(\mathrm{A} \cup \mathrm{B}) \supseteq \mathrm{D_d}(\mathrm{A}) \cup \mathrm{D_d}(\mathrm{B})$. Let $x \in D_d(A \cup B)$. We show that $x \in D_d(A) \cup D_d(B)$. Suppose $x \notin D_d(A) \cup D_d(B)$. Then $x \notin D_d(A)$ and $x \notin D_d(B)$. Therefore, x is neither a differential limit point of A nor of B. Hence, there exist differential neighborhoods M and N of x such that M contains no points A other than x and N contains no points B other than x. Since M and N are differential neighborhood of x, by Proposition 2.3 we have $M \cap N$ is differential neighborhood of x. Also, $M \cap N$ contains no point of $A \cup B$ other than x. Therefore, x is not a differential limit point of $A \cup B$. Hence, $x \notin D_d(A \cup B)$. This is a contradiction, since $x \in D_d(A \cup B)$. Therefore, $x \in D_d(A) \cup D_d(B)$. Hence, $D_d(A \cup B) \subseteq D_d(A) \cup D_d(B)$. Thus, $\mathrm{D_d}(\mathrm{A} \cup \mathrm{B}) = \mathrm{D_d}(\mathrm{A}) \cup \mathrm{D_d}(\mathrm{B})$.

iii. We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then by i, $\mathrm{D_d}(\mathrm{A} \cap \mathrm{B}) \subseteq \mathrm{D_d}(\mathrm{A})$ and $\mathrm{D_d}(\mathrm{A} \cap \mathrm{B}) \subseteq \mathrm{D_d}(\mathrm{B})$. Therefore, $\mathrm{D_d}(\mathrm{A} \cap \mathrm{B}) \subseteq \mathrm{D_d}(\mathrm{A}) \cap \mathrm{D_d}(\mathrm{B})$.

Conclusion

This work is devoted to introducing and discussing the concepts of differential limit points of a set with respect to differential open sets. We have established the properties of differential neighborhoods and differential limit points in derivative topological spaces.

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