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Total Geo Chromatic Number for Degree Splitting Graphs of Certain Graphs

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Abstract: A total geo chromatic set of a graph G is a geo chromatic set S_c such that the subgraph induced by S_c has no isolated vertices. The minimum cardinality of a total geo chromatic set of G is the total geo chromatic number of G and is denoted by $\chi_{tg}(G)$. A total geo chromatic set of cardinality $\chi_{tg}(G)$ is called a χ_{tg}-set of G. The total geo chromatic number of some standard graphs are determined and some general properties satisfied by this concept are studied. In this paper, we investigate the total geo chromatic number of degree splitting graphs of certain graphs.

Keywords: *Geodetic number, chromatic number, geo chromatic number, total geo chromatic number, degree splitting graphs.*

AMS: 05C12

1. Introduction

Let $G = (V, E)$ be a finite undirected connected graph without multiple edges or loops. The order and size of G are denoted by k and l respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices p and q in a connected graph G, the distance $d(p,q)$ is the length of a shortest $p-q$ path in G. An $p-q$ path of length $d(p,q)$ is called an $p-q$ geodesic. A vertex x is said to lie on an $p-q$ geodesic P' if x is a vertex of P' including the vertices of p and q. The neighborhood of a vertex x is the set $N(x)$ consisting of all vertices y which are adjacent with x. A vertex x is an extreme vertex of G if the subgraph induced by its neighbors is complete.

The closed interval $I[p,q]$ consists of all vertices lying on some $p-q$ geodesic of G, while for $S \subseteq V$, $I[S] = \cup_{p,q \in S} I[p,q]$. If $I[S] = V$, then a set S of vertices is a geodetic set and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic number of a graph was introduced in [3,4] and further studied in [5,7].

The concept of geo chromatic number was introduced by S. B. Samli and S. R. Chellathurai in [1] and further studied in [2,9]. A geodetic set S is said to be a geo chromatic set S_c of G, if S is both a geodetic set and a chromatic set of G. The minimum cardinality of a geo chromatic set of G is the geo chromatic number of G and is denoted by $\chi_{gc}(G)$.

The concept of geo chromatic set of G has motivated us to introduce the new geo chromatic set conception of totoal geo chromatic set. We call the minimum cardinality of a totoal geo chromatic set of G, the total geo chromatic number of G.

Definition 1.1[10]

A subset $S_c \subseteq V(G)$ is said to be total geo chromatic set of G if S_c is both a total geodetic and a chromatic set of G. The minimum cardinality of a total geo chromatic set is the total geo chromatic number of G and is denoted by $\chi_{tg}(G)$. A total geo chromatic set of minimum cardinality is called a χ_{tg}- set of G.

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2. Preliminaries

Theorem 2.1. *[10] Every extreme vertex of a connected graph G belongs to every total geo chromatic set of G.*

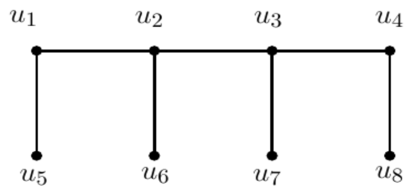
Theorem 2.2. *[10] Every support vertex of a connected graph G belongs to every total geo chromatic set of G.*

3 Degree splitting graphs of some known graphs and their total Geo Chromatic Number

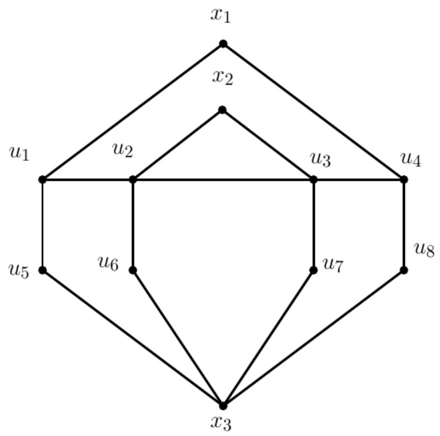
Definition 3.1 [11]: Let $G = (V, E)$ be a connected graph with $V(G) = S_1 \cup S_2 \cup \ldots \cup S_r \cup R$, where S_i is the set having at least two vertices of some degree and $R = V(G) - \cup S_i$ for $1 \leq i \leq r$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $x_1, x_2, \ldots, x_r$ and joining x_i to each vertices in S_i for $1 \leq i \leq r$.

Example 3.2

Consider a connected graph G and their corresponding degree splitting graph $DS(G)$ given in Figure 1.



G



$DS(G)$

Figure 1

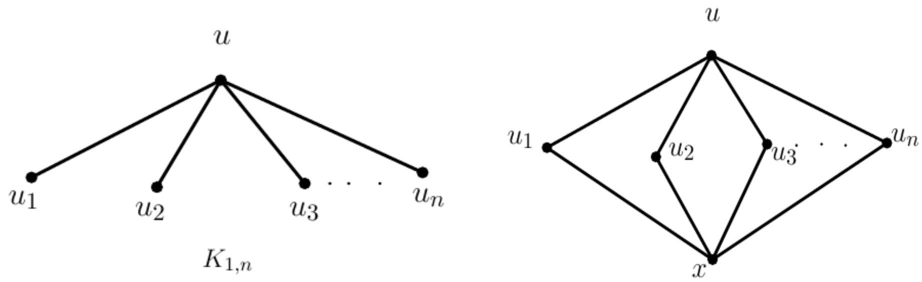
Here $S_1 = \{u_1, u_4\}, S_2 = \{u_2, u_3\}, S_3 = \{u_5, u_6, u_7, u_8\}$ and $R = \emptyset$.

Theorem 3.3

For integer $n \geq 2, \chi_{tg}\left(DS(K_{m,n})\right) = 3$.

Proof.

Let $u_1, u_2, \dots, u_n$ are the pendant vertices and u is the full degree vertex of the star graph $K_{1,n}$ and x be the corresponding vertex which is added to obtained the graph $DS(K_{1,n})$. Then $V\left(DS(K_{1,n})\right) = \{u, u_1, u_2, \dots, u_n, x\}$ and so $\left|V\left(DS(K_{1,n})\right)\right| = n + 2$.



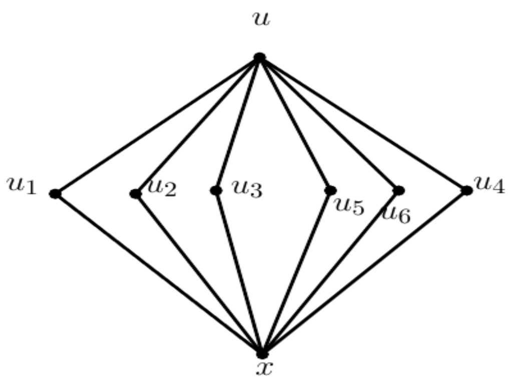
$K_{1,n}$ $DS(K_{1,n})$

Figure 2

Since $DS(K_{1,n})$ is connected, we have $2 \leq \chi_{tg}\left(DS(K_{1,n})\right) \leq n + 2$. Consider $S = \{u, x\}$. Then, there are $n-$geodesic path which travels between u and x, that includes all the vertices of $DS(K_{1,n})$. Therefore $S = \{u, x\}$ is a geodetic set of minimum cardinality. But if we splitted the degree splitting graph $DS(K_{1,n})$ in to two partition $V_1 = \{u, x\}$ and $V_2 = \{u_1, u_2, \dots, u_n\}$. Clearly the chroamtic number of $DS(K_{1,n})$ is two. As we colour c_1 to each vertex in V_1 and colour c_2 assigns to each verices in V_2, that S is not a chromatic set of $DS(K_{1,n})$. It follows that S is not a total geo chromatic set of $DS(K_{1,n})$. Thus, $\chi_{tg}\left(DS(K_{1,n})\right) > |S| = 2$. Now, it is clear that $S_i = \{u, u_i, x\}, 1 \leq i \leq n$ is a total geo chromatic set of $DS(K_{1,n})$ and so $\chi_{tg}\left(DS(K_{1,n})\right) \leq |S_i| = 3$. Hence, $\chi_{tg}\left(DS(K_{1,n})\right) = 3$.

Example 3.4

Consider the connected graph $DS(K_{1,6})$ in Figure 3. By Theorem 3.3, $S = \{u, u_1, x\}$ is a minimum total geo chromatic set of $DS(K_{1,6})$ and hence $\chi_{tg}\left(DS(K_{1,6})\right) = |S| = 3$.



$DS(K_{1,6})$

Figure 3

Here $S_1 = V(K_{1,6})$ is the unique total geo chromatic set of $K_{1,6}$ and so $\chi_{tg}(K_{1,6}) = 7$.

Thus the total geo chromatic set of G and $S'(G)$ are different.

Remark 3.5

Every extreme vertex of G need not belong to every minimum total geo chromatic set of $DS(G)$.

For example, we consider the graph which is given in Figure 4.

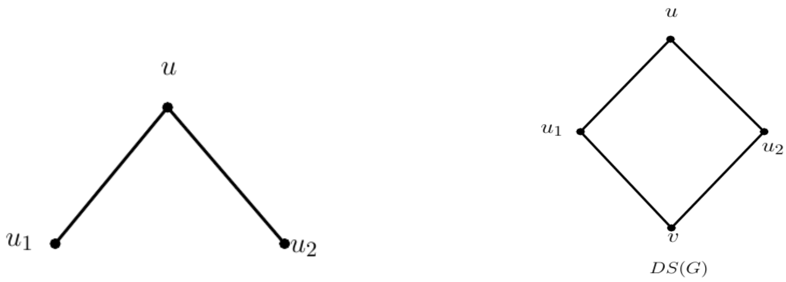


Figure 4

Here $S_1 = \{u, u_1, x\}$ is a minimum total geo chromatic set of $DS(G)$. But u_2 is an extreme vertex of G, which does not belong to S.

It is to be noted that u_1 is an extreme vertex of G, belong to S. But u_1 and u_2 both does not in S.

Observation 3.6

For any connected graph G, there is no obvious relation connecting $\chi_{tg}(G)$ and $\chi_{tg}(DS(G))$.

Example for $\chi_{tg}(DS(G)) < \chi_{tg}(G)$.

Let us consider the connected graph G in Figure 5.

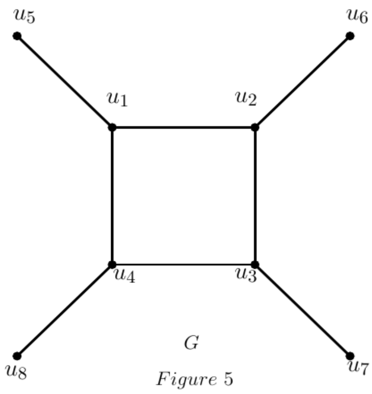
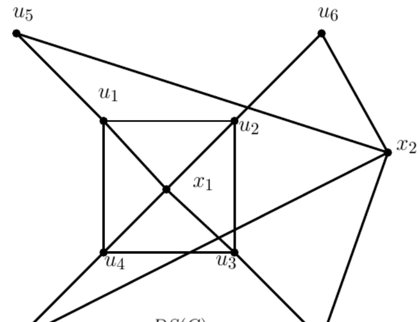


Figure 5

Here $S = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ is the unique total geo chromatic set of G and so $\chi_{tg}(G) = 8$.

Now we degree split every vertex of G, is obtained a new connected graph $DS(G)$ and is given in Figure 6.



$DS(G)$

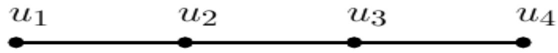
Figure 6

Here $S_1 = \{x_1, x_2, u_2, u_6\}$ is a minimum total geo chromatic set of $DS(G)$ and so $\chi_{tg}(DS(G)) = 4$.

In this case, $\chi_{tg}(DS(G)) < \chi_{tg}(G)$.

Example for $\chi_{tg}(DS(G)) = \chi_{tg}(G)$.

Consider the connected graph G in Figure 7.

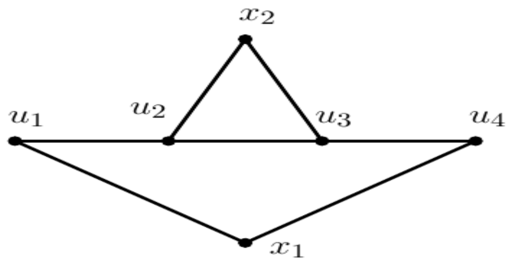


G

Figure 7

Here $S = \{u_1, u_2, u_3, u_4\}$ is the unique total geo chromatic set of G and so $\chi_{tg}(G) = 4$.

Now we degree split every vertex of G, to obtained a new connected graph $D(G)$ and is given in Figure 8.



$DS(G)$

Figure 8

Here $S_1 = \{u_1, u_2, x_1, x_2\}$ is a minimum total geo chromatic set of $DS(G)$ and so $\chi_{tg}(DS(G)) = 4$.

Thus in this case $\chi_{tg}(DS(G)) = \chi_{tg}(G)$. Example for $\chi_{tg}(DS(G)) > \chi_{tg}(G)$.

Consider the connected graph G, in Figure 9.

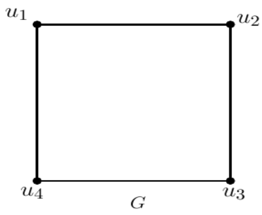


Figure 9

Here, $S_1 = \{u_1, u_2, u_3\}$ is a minimum total geo chromatic set of G and so, $\chi_{tg}(G) = 3$.

Now we degree split every vertex of G, to obtained a new connected graph $DS(G)$ and is given in Figure 10.

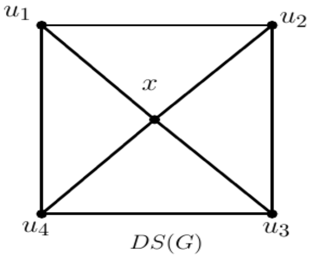


Figure 10

Here $S_1 = \{u_1, u_2, u_3, x\}$ is a minimum total geo chromatic set o f $DS(G)$ and so, $\chi_{tg}(DS(G)) = 4$.

In this case $\chi_{tg}(DS(G)) > \chi_{tg}(G)$.

Remark 3.7.

Every support vertex of G belong to every total geo chromatic set of $DS(G)$.

Theorem 3.8.

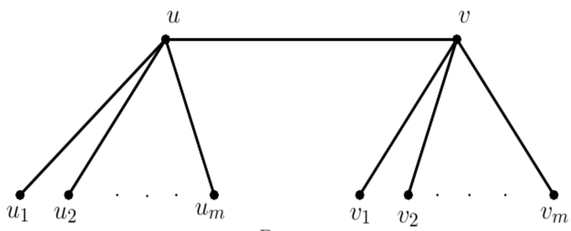
For the bistar graph $B_{m,n}$ $(m \geq n)$, $\chi_{tg}\left(DS(B_{m,n})\right) = 4$.

Proof.

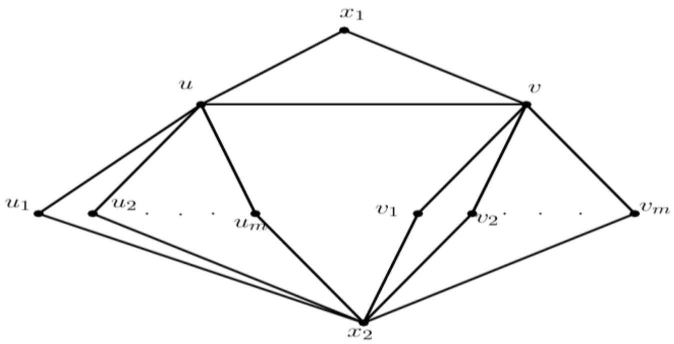
Consider the bistrar $B_{m,n}$ with vertex set $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Here each u_i and v_j are the non-adjacent vertices adjacent to u and v, respectively. To obtain the graph $DS(B_{m,n})$, two cases where arise:

Case (i) $m = n$.

Let x_1 and x_2 be the corresponding vertices which are added to obtain $DS(B_{m,n})$. Then, $V\left(DS(B_{m,n})\right) = \{u, v, u_i, v_i, x_1, x_2; 1 \leq i \leq m\}$ and so $\left|V\left(DS(B_{m,n})\right)\right| = 2m + 4$. This graph is given in Figure 11.



$B_{m,m}$



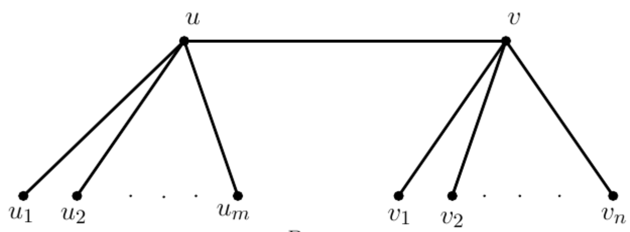
$DS(B_{m,m})$

Figure 11

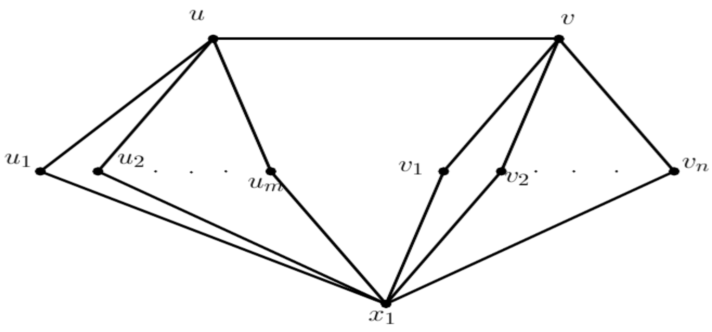
Since $DS(B_{m,n})$ is connected, $\chi_{tg}\left(DS(B_{m,n})\right) \geq 2$. Consider $I[x_1, x_2]$, which has $2m$ trasversal geodesic path of length three between x_1 and x_2, which include all the vertices $DS(B_{m,n})$. Therefore $S = \{x_1, x_2\}$ is a geodetic set of minimum cardinality. But to assign colour to each vertex as a proper colouring we need minimum three vertices because the graph $DS(B_{m,n})$ has partition as $V_1 = \{x_1, x_2\}, V_2 = \{u_1, u_2, \dots, u_m, v\}$ *and* $V_3 = \{v_1, v_2, \dots, v_m, u\}$. Clearly $V\left(DS(B_{m,n})\right) = V_1 \cup V_2 \cup V_3$ and $V_1 \cap V_2 \cap V_3 = \emptyset$. Therefore that S is not a chromatic set of $DS(B_{m,n})$ and so $\chi_{tg}\left(DS(B_{m,n})\right) \geq 3$. To be geodetic set with minimum cardinality, we must include exactly one vertex from each $V_j, 1 \leq j \leq 3$. If we consider $S = \{x_1, x_2, u, u_1\}$, then it is clear that S is a total geodetic set as well as a chromatic set of minimum cardinality. Hence, that $S = \{x_1, x_2, u, u_1\}$ is a minimum total geo chromatic set of $DS(B_{m,n})$ and so $\chi_{tg}\left(DS(B_{m,n})\right) = |S|$=4.

Case (ii) $m \neq n$.

Then clearly $m > n$. Let x_1 be the corresponding vertex which is added to obtain $DS(B_{m,n})$. Then clearly $V\left(DS(B_{m,n})\right) = \{u, v, u_i, v_j, x_1; 1 \leq i \leq m, 1 \leq j \leq n\}$ and hence $\left|V\left(DS(B_{m,n})\right)\right| = m + n + 3$. This graph is shown in Figure 12.



$B_{m,n}$

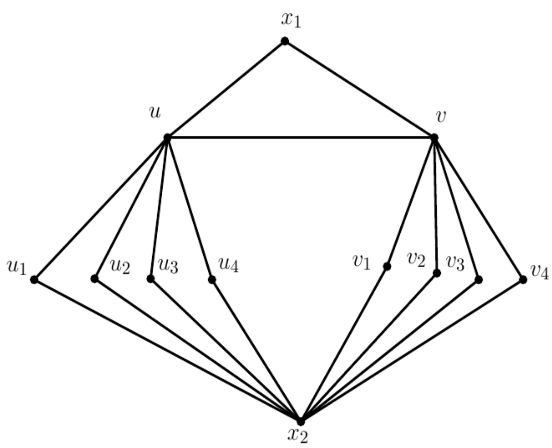


$DS(B_{m,n})$
Figure 12

Consider $I[u, x_1]$, which has m geodesic path of length two betweeen u and x_1, which include all the vertices adjaccent to u except v in $DS(B_{m,n})$. Simillarly, $I[v, x_1]$, which has n geodesic path of length two between v and x_1, which include all the vertices adjacent to v except u in $DS(B_{m,n})$. Therefore, if $S = \{u, v, x_1\}$, then it is clear that $I[S] = V\left(DS(B_{m,n})\right)$ and so that S is a geodetic set of $DS(B_{m,n})$. But the subgraph induced by S has isolated vertices. It follows that S is not a total geodetic set of $DS(B_{m,n})$. Since x_1 is adjacent with $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$, if we select $S = \{u, v, u_1, x_1\}$, then clearly S is a total geodetic set of $DS(B_{m,n})$. As similar in case(i), to be a chromatic set with minimum cardinality, we must include exactly one vertex from each $V_j, 1 \leq j \leq 3$. Therefore, that $S = \{u, v, u_1, x_1\}$ is a total geo chromatic set of minimum cardinality and hence $\chi_{tg}\left(DS(B_{m,n})\right) = |S| = 4$.

Example 3.9

Consider the connected graph $DS(B_{4,4})$ is given in Figure 13, By Theorem 3.8, $S = \{x_1, x_2, u, u_1\}$ is a minimum total geo chromatic set of $DS(B_{4,4})$ and hencce $\chi_{tg}\left(DS(B_{4,4})\right) = |S| = 4$.



$DS(B_{4,4})$
Figure 13

Theorem 3.10.

For integer $m, n \geq 2, \chi_{tg}\left(DS(B_{m,n})\right) = \begin{cases} 5 \; if \; m = n \\ 4 \; otherwise \end{cases}$.

Proof.

Consider the graph $K_{m,n}$ with vertex set $V(K_{m,n}) = \{u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ with bipartition $X = \{u_1, u_2, \dots, u_m\}$ and $Y = \{v_1, v_2, \dots, v_n\}$. To obtain $DS(K_{m,n})$, we consider two cases.

Case (i) $m = n$

This case we see that each vertex is of same degree and let x_1 be the added vertex, which is now adjacent to each u_i and v_j for $1 \leq i \leq m$ and $1 \leq j \leq n$. The graph $DS(K_{m,n})$ is obtained and is given in Figure 14.

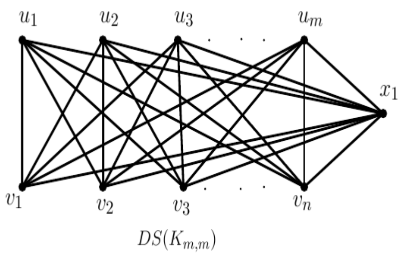
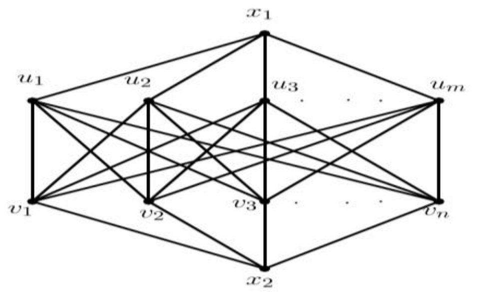


Figure 14

Clearly $V\left(DS(K_{m,n})\right) = \{u_i, v_j, x_1; 1 \leq i, j \leq m\}$ and so $\left|V\left(DS(K_{m,n})\right)\right| = 2m + 1$. Consider the set $S = \{u_1, u_2, v_1, v_2\}$. Since x is adjacent to other vertices if $DS(K_{m,n})$, that S is a total geodetic set $DS(K_{m,n})$. By assigning the proper colouring as $c(u_1) = c(u_2) = \cdots = c(u_m) = 1, c(v_1) = c(v_2) = \cdots = c(v_n) = 2$ and $c(x_1) = 3$. Therefore the vertices in X assign colour 1, the vertices in Y assign colour 2 and x_1 assigns colour 3. It follows that S is not a chromatic set of $DS(K_{m,n})$ and so that S is not a total geo chromatic set of $DS(K_{m,n})$. Now it is clear that $S = \{u_1, u_2, v_1, v_2, x_1\}$ is a total geo chromatic set of minimum cardinality. Hence $\chi_{tg}(K_{m,m}) = 5$.

Case (ii) $m \neq n$.

This case, it is clear that every vertices in X is of same degree and every vertices in Y is of same degree with $\deg(u_i) \neq \deg(v_j)$ for every $u_i \in X$ and $v_j \in Y$. So let x_1 and x_2 be the vertices corresponding to addee the vertices of X and Y, respectively. Then $V\left(DS(K_{m,n})\right) = \{u_i, v_j, x_1, x_2; 1 \leq j \leq m, 1 \leq i \leq n\}$ and so $\left|V\left(DS(K_{m,n})\right)\right| = m + n + 2$. This connected graph is given in Figure 15.



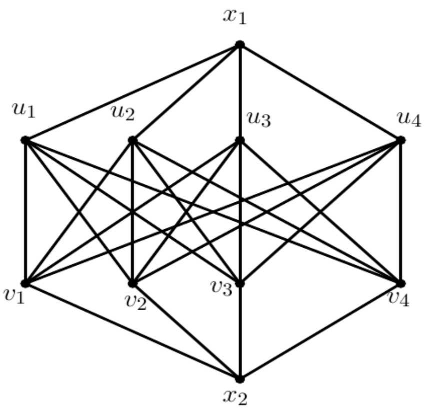
$DS(K_{m,n})$
Figure 15

Now consider the set $S = \{x_1, x_2\}$, where $I[S] = V\left(DS(K_{m,n})\right)$ and that S contains vertices x_1, x_2, which are not adjacent. Therefore that S is not a total geodetic set of $DS(K_{m,n})$. Also, this graph has partition X, Y and x_1 include with Y and x_2 include with Y. Therefore, two colour is enough to assign a proper colouring. For a chromatic set we need one vertex from $X \cup \{x_2\}$ and one vertex from $Y \cup \{x_1\}$. It follows that S is a chromatic set of minimum cardinality. Since $< S >$ has isolated vertices, that S itself is not a total geo chromatic set of $DS(K_{m,n})$. Since $d(x_1, x_2) = 3$, there must be two internal vertices in $d(x_1, x_2)$. It is clear that $S = \{x_1, u_1, v_1, x_2\}$ is a total geo chromatic set of minimum cardinality. Hence $\chi_{tg}\left(DS(B_{m,n})\right) = |S| = 4$.

Thus $\chi_{tg}\left(DS(B_{m,n})\right) = \begin{cases} 5 \text{ } if \text{ } m = n \\ 4 \text{ } otherwise \end{cases}$.

Example 3.11.

Consider the connected graph $DS(K_{4,3})$ given in Figure 16. By Theorem 12, $S = \{x_1, u_1, v_1 x_2\}$ is a minimum total geo chromatic set of $DS(K_{4,3})$ and hence $\chi_{tg}\left(DS(K_{4,3})\right) = |S| = 4$.



$DS(K_{4,3})$
Figure 16

References

[1] S. BEULAH SAMLI AND S. ROBINSON CHELLATHURAI: Geo Chromatic Number of a Graph, Int. J. Sci. Res. Math. and Stat. Scien., 5(6)(2018), 259 – 264.

[2] S. BEULAH SAMLI, J. JOHN AND S. ROBINSON CHELLATHURAI: The Double Geo Chromatic Number of a Graph, Bull. Int. Math. Virtual Inst., 11(1)(2021), 55 – 68.

[3] F. BUCKLEY AND F. HARARY: Distance in Graphs, Addison – Wesley, Redwood City, CA, 1990.

[4] G. CHARTRAND, F. HARARY AND P. ZHANG: On the Geodetic Number of a Graph, Networks 39(2002), 1 – 6.

[5] H. ESCUADRO, R. GERA, A.HANSBERG, N. JAFARI RAD AND L. VOLKMANN : Geodetic Domination in Graphs, J. Combin. Math. Combin. Comput., 77(2011), 89 – 101.

[6] F. HARARY: Graph Theory Addison – Wesley, Reading, Mass, 1972.

[7] F. HARARY, E. LOUKAKIS AND C. TSOUROS: The Geodetic Number of a Graph Math. Comput. Modeling, 17(11) (1993), 89 – 95.

[8] D. A. MOJDEH AND N. J. RAD: Connected Geodomination in Graphs J. Discrete Math. Scien, and Cryp., 9(1) (2006), 177 – 186.

[9] M. MOHAMMED ABDUL KHAYOOM AND P. ARUL PAUL SUDHAHAR: Monophonic Chromatic Parameter in a Connected Graph Int. J. Math. Anal., 11(19)(2017), 911 – 920.

[10] S. ANNIE AJILA AND J. ROBERT VICTOR EDWARD: The total geo chromatic number of a graph, Malaya Journal of Matematik, 8(4)(2020), 2337-2341.
https://doi.org/10.26637/MJM0804/0178

[11] R. PONRAJ, S. SOMASUNDARAM, On the degree splitting graph of a graph, National Academy Science Letters, 27(7-8), 275-278, (2004)

[12] C. JAYASEKARAN AND S.V. ASHWIN PRAKASH, Detour Global Domination for Degree Splitting graphs of some graphs, Ratio Mathematica, 47 (2023).