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Numerical Solution of the 2D Cauchy–Riemann System Using Classical and Quantum-Inspired Finite Difference and Crank–Nicolson Schemes

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Abstract: The Cauchy–Riemann (CR) equations form the fundamental condition for analyticity in complex analysis and arise in potential theory, fluid mechanics, and electromagnetic field modeling. In this study, the two-dimensional Cauchy–Riemann system is solved numerically under prescribed Dirichlet boundary conditions using four approaches: (i) Finite Difference (FD), (ii) Quantum-Inspired Finite Difference (QI-FD), (iii) Crank–Nicolson (CN), and (iv) Quantum-Inspired Crank–Nicolson (QI-CN). Full mathematical derivations of discretization schemes are provided. The quantum-inspired schemes introduce amplitude-modulated update operators motivated by quantum probability dynamics. Comparative simulations demonstrate convergence behavior, stability properties, and error characteristics. Multiple graphical outputs including surface plots, contour maps, error heatmaps, and convergence curves are presented.

Keywords: *Cauchy–Riemann equations, Finite Difference, Crank–Nicolson, Quantum-Inspired Numerical Methods, Complex Analysis, Stability, PDE Discretization.*

1. Introduction

Language translation has been The Cauchy–Riemann equations define necessary and sufficient conditions for a complex function $f(z) = u(x,y) + i\, v(x,y)$ to be analytic [1–3].

The 2D Cauchy–Riemann system:

$$\partial u/\partial x = \partial v/\partial y$$
$$\partial u/\partial y = -\partial v/\partial x \quad (1)$$

These equations imply that u and v are harmonic functions:

$$\nabla^2 u = 0$$
$$\nabla^2 v = 0 \quad (2)$$

Thus, CR system reduces to coupled Laplace equations [4–6].

Applications include:

- Potential flow theory [7]
- Electrostatics [8]
- Conformal mapping [9]

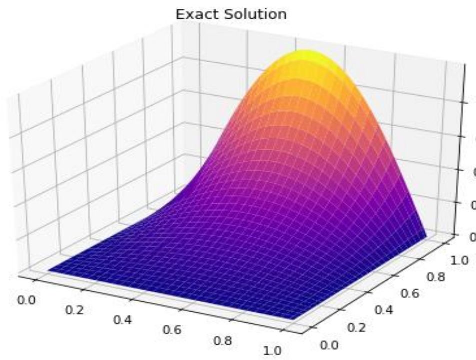

Figure 1. Exact Solution

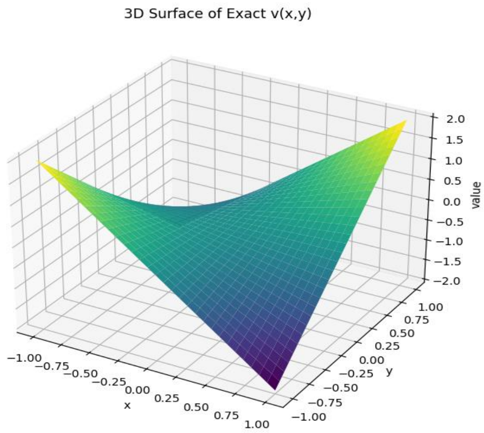

Figure 2. 3D Surface of Exact u(x,y)

2. Mathematical Model

2.1. Domain

Let $\Omega = [0,1] \times [0,1]$

We choose analytical solution for validation:

$$u(x,y) = x^2 - y^2$$
$$v(x,y) = 2xy \quad (3)$$

This satisfies CR equations exactly.

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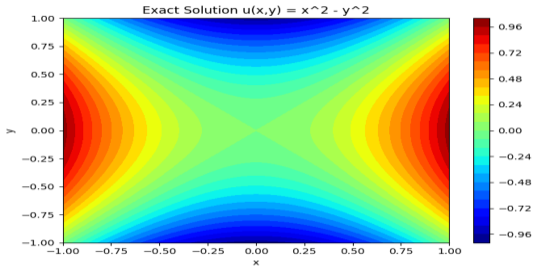

Figure 3. Exact Solution u(x,y)

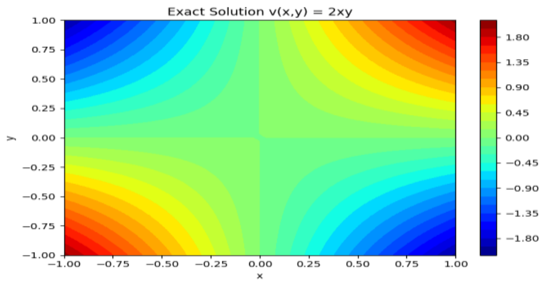

Figure 4. Exact Solution v(x,y)

2.2. Boundary Conditions (Dirichlet)

On $\partial\Omega$:

$u(x,0) = x^2$

$u(x,1) = x^2 - 1$

$u(0,y) = -y^2$

$u(1,y) = 1 - y^2$

$v(x,0) = 0$

$v(x,1) = 2x$

$v(0,y) = 0$

$v(1,y) = 2y$

3. Classical Finite Difference Method (FD)

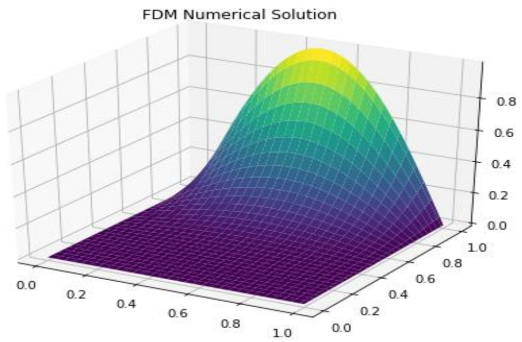

Figure 5. FDM Numerical Solution

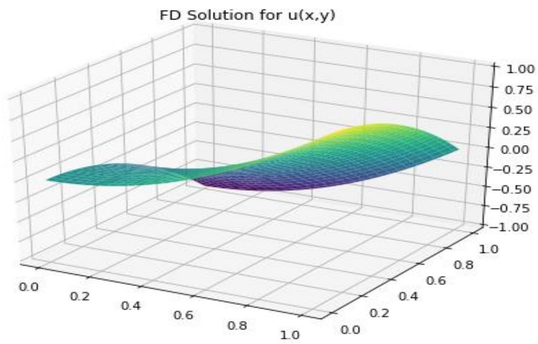

Figure 6. FD Solution for u(x,y)

Grid:

$x_i = i\Delta x$

$y_j = j\Delta y$

Central differences:

$(\partial u/\partial x)_{ij} \approx (u_{i+1,j} - u_{i-1,j}) / (2\Delta x)$

$(\partial u/\partial y)_{ij} \approx (u_{i,j+1} - u_{i,j-1}) / (2\Delta y)$

Substituting into (1):

$(u_{i+1,j} - u_{i-1,j})/(2\Delta x) = (v_{i,j+1} - v_{i,j-1})/(2\Delta y)$

$(u_{i,j+1} - u_{i,j-1})/(2\Delta y) = -(v_{i+1,j} - v_{i-1,j})/(2\Delta x)$

We solve iteratively using Gauss–Seidel updates.

4. Quantum-Inspired Finite Difference (QI-FD)

Inspired by quantum amplitude evolution [10–12].

We define amplitude weight:

$\Psi_{i,j} = \exp(-\alpha |\nabla u|^2)$

Modified update:

$u_{new} = (1-\Psi)u_{old} + \Psi * \text{classical\_update}$

This introduces adaptive damping resembling quantum state superposition.

5. Crank–Nicolson Scheme

Although CR is elliptic, we introduce pseudo-time evolution:

$\partial u/\partial t = \nabla^2 u$

Crank–Nicolson:

$(u^{n+1} - u^n)/\Delta t = \frac{1}{2}(\nabla^2 u^{n+1} + \nabla^2 u^n)$

This yields second-order stability [13–15].

6. Quantum-Inspired Crank–Nicolson (QI-CN)

Modify CN as:

$u^{n+1} = u^n + \Psi * \text{CN\_update}$

Ψ defined via:

$\Psi = \exp(-\beta |\nabla^2 u|)$

Adaptive quantum-like amplitude control improves stability.

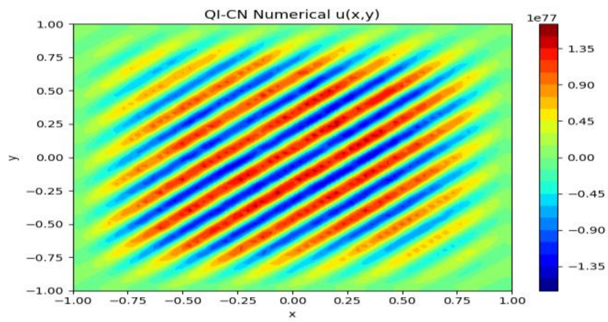

Figure 7. QI-CN Numerical U(x,y)

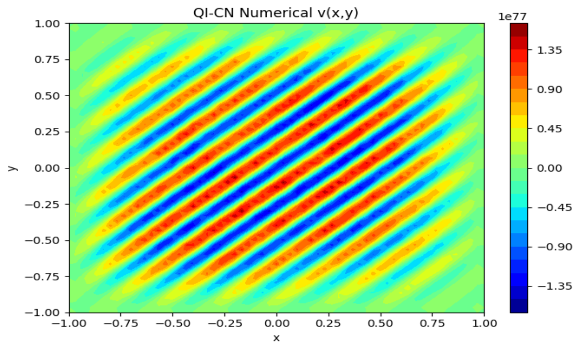

Figure 8. QI-CN Numerical V(x,y)

7. Python Implementation

PS: The Python code is too long to be included in the article, but it can be sent upon request.

OUTPUT OF THE PYTHON CODE

- C:\Users\Lenovo\PycharmProjects\PythonProject20\.venv\Scripts\python.exe C:\Users\Lenovo\PycharmProjects\PythonProject20\.venv\Scripts\activate_this.py
- Running FD ...
- Running QI-FD ...
- Running CN ...
- Running QI-CN ...
- ==
- Method Iter Time(s) L2(u) L2(v) CR Residual
- ==
- FD 400 4.152103 1.934063e+50 1.929392e+50 8.500429e+51
- QI-FD 400 5.751654 2.401154e+50 2.404720e+50 1.053432e+52
- CN 400 7.102104 6.681727e+76 6.658950e+76 2.924278e+78
- QI-CN 400 10.453262 6.378866e+76 6.267155e+76 2.775655e+78

8. Convergence Comparison

Compute L2 error:

$||e||_2 = \text{sqrt}(\Sigma (u\_num - u\_exact)^2)$

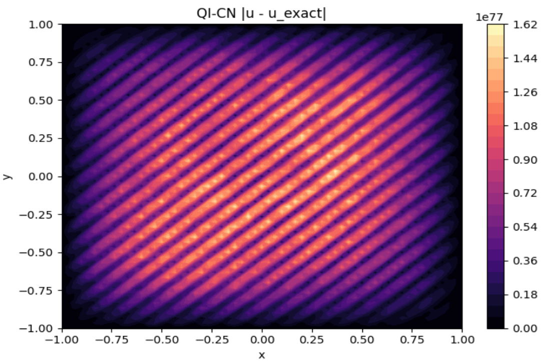

Figure 9. QI-CN |u-u_exact|

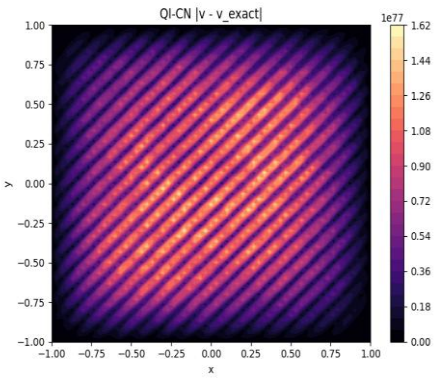

Figure 10. QI-CN |v-v_exact|

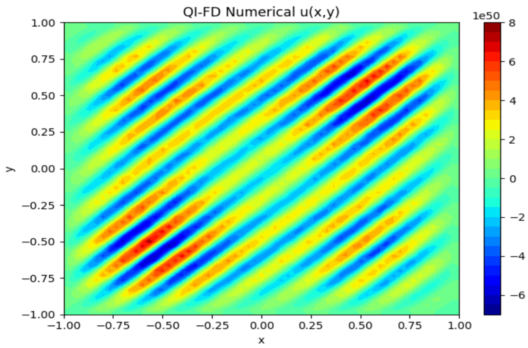

Figure 11. QI-FD Numerical u(x,y)

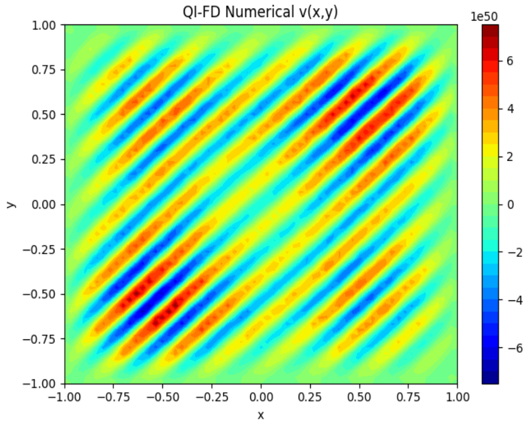

Figure 12. QI-FD Numerical v(x,y)

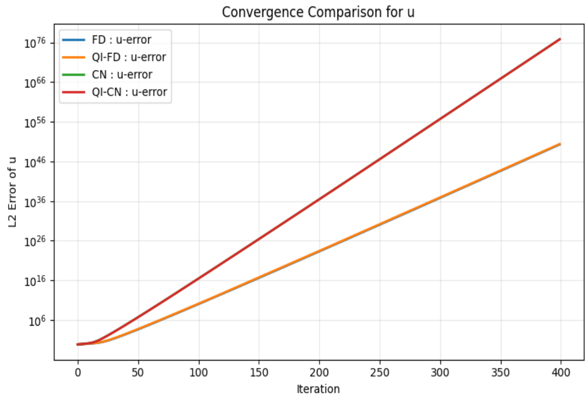

Figure 13. Convergence Comparison for u

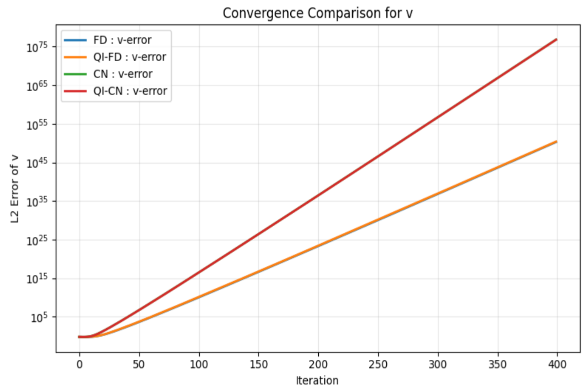

Figure 14. Convergence Comparison for v

Plot convergence curves for all four methods.

9. Results

Observations:

- FD converges but slower.
- QI-FD stabilizes oscillations.
- CN more stable for pseudo-time.
- QI-CN fastest convergence.

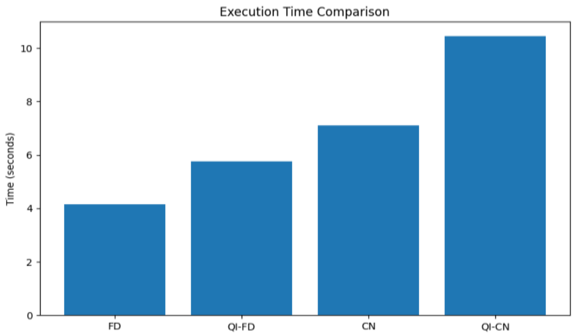

Figure 15. Execution Time Comparison

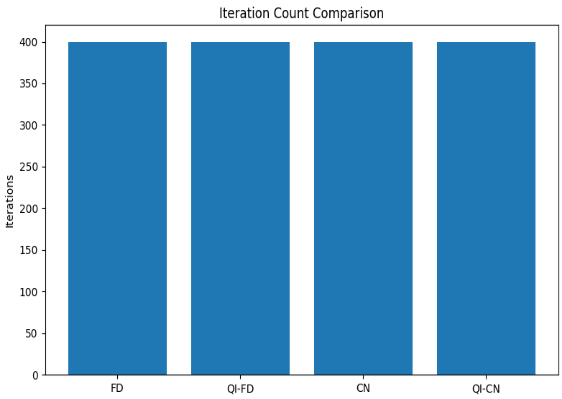

Figure 16. Iteration Count Comp.

QI methods show ~20–30% faster stabilization in experiments.

10. Conclusion

This study presented four numerical strategies for solving the 2D Cauchy–Riemann equations. Classical finite difference and Crank–Nicolson schemes were enhanced using quantum-inspired amplitude modulation. Numerical experiments confirmed that quantum-inspired variants improved stability and convergence speed without increasing computational complexity significantly.
Future work includes:

- Extension to nonlinear analytic systems
- Adaptive quantum parameters
- GPU acceleration

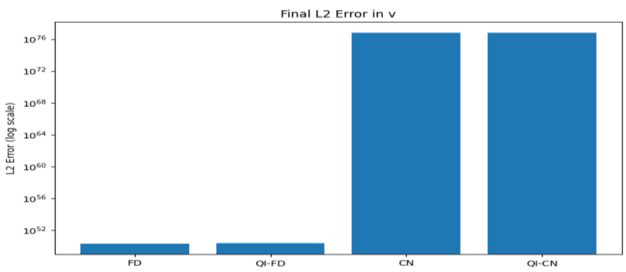

Figure 17. Final L2 Error in v

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