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On Hub Domination Number of Transformation Graphs

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Abstract: For a graph G, a set $H \subseteq V$ is a hub dominating set if every vertex u in $V - H$ is adjacent to a vertex $v \in H$ and any two non-adjacent vertices u, w in $V - H$ has a path in G in which all the internal vertices of the path must be in H. The least cardinality taken over all hub dominating set H of vertices in graph G is known as the hub domination number. We denote the hub domination number of G by $\gamma_h(G)$.

Keywords: *Hub dominating set, Hub domination number, Transformation graph.*

AMS Subject Classification: 05C38, 05C69, 05C07.

1. Introduction

Domination is an important topic in graph theory and has many applications in communication networks and optimization problems. The basic concepts of graph theory can be found in Harary [2]. The study of domination was introduced by Ore [3]. The concept of hub in graph was first proposed by Walsh in 2006 [4]. Hub domination has evolved by combining graph domination and hub concepts. In 2001, Wu and Meng introduced transformation graphs G^{xyz} of G and investigated some basic properties of eight types of transformation graphs of a simple graph G [5]. In this paper we studied the hub domination number of some transformation graphs.

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2. Preliminaries

Definition 2.1 Let $G = (V(G), E(G))$ be a graph and x, y, z be three variables taking value $+$ or $-$. The transformation graph G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set, and for $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds:

(a) For $\alpha, \beta \in V(G)$, α and β are adjacent in G if $x = +$; α and β are not adjacent in G if $x = -$.

(b) For $\alpha, \beta \in E(G)$, α and β are adjacent in G if $y = +$; α and β are not adjacent in G if $y = -$.

(c) For $\alpha \in V(G)$ and $\beta \in E(G)$, α and β are incident in G if $z = +$; α and β are not incident in G if $z = -$.

Definition 2.2 The transformation graph G^{+++} of G is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows:

(a) Two elements in $V(G)$ are adjacent if and only if they are adjacent in G.

(b) Two elements in $E(G)$ are adjacent if and only if they are adjacent in G.

(c) One element in $V(G)$ and one element in $E(G)$ are adjacent if and only if they are incident in G.

3. Hub Domination Number Of Transformation Graphs

Definition 3.1 For a graph G, a set $H \subseteq V$ is a hub dominating set if every vertex u in $V-H$ is adjacent to a vertex v in H and any two non-adjacent vertices u, w in $V-H$ has a path in G in which all the internal vertices of the path must be in H. The least cardinality taken over all hub dominating set H of vertices in graph G is known as the hub domination number. We denote the hub domination number by $\gamma_h(G)$.

Theorem 3.2

For all $n \geq 3$, $\gamma_h(P_n^{+++}) = n-2$, where P_n^{+++} is the transformation graph of the Path graph P_n.

Proof.

Consider the transformation graph of Path P_n^{+++} with $2n-1$ vertices.

Let $V(P_n^{+++}) = \{v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_{n-1}\}$ where v_1, v_n are of degree 2; v_i's where $2 \leq i \leq n-1$ are of degree 4; e_1, e_n are of degree 3 and e_j's where $2 \leq j \leq n-1$ are of degree 4. We claim $H = \{v_2, v_3, \dots v_{n-1}\}$ is a hub dominating set. Clearly v_1, v_n are adjacent to v_2 and v_{n-1} respectively. Also, every e_j where $1 \leq j \leq n-2$ are adjacent to v_{i+1} and e_{n-1} is adjacent to v_{n-1}. For any pair of non-adjacent vertices v_1, v_n there exists a path $v_1 v_2 \dots v_n$ in P_n^{+++}. Also, for any pair of non-adjacent vertices e_i, e_j where $i < j$ and $1 \leq i, j \leq n-1$, there exists a path $e_i v_{i+1} v_{i+2} \dots v_j e_j$ in P_n^{+++}. Hence, H is a hub dominating set with $n-2$ vertices. Therefore, $\gamma_h(P_n^{+++}) = n-2, \forall\, n \geq 3$.

Remark 3.3

$\gamma_h(P_n^{+++}) = 1$, when $n = 2$.

Since, P_2^{+++} is a Complete graph K_3, $\gamma_h(P_2^{+++}) = 1$.

Theorem 3.4

For all $n \geq 3, \gamma_h(C_n^{+++}) = n-1$, where C_n^{+++} is the transformation graph of the Cycle graph C_n.

Proof.

Consider the transformation graph of Cycle C_n^{+++} with $2n$ vertices. Let $V(C_n^{+++}) = \{v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_n\}$, v_i, e_j where $1 \leq i, j \leq n$ are of degree 4. We claim $H = \{v_1, v_2, \dots v_{n-1}\}$ is a hub dominating set. Clearly v_n is adjacent to v_1 and every e_i's are adjacent to v_i's where $1 \leq i \leq n$. For any pair of non-adjacent vertices
$v_n, e_j \in V(C_n^{+++}) - H$, there exists a path $v_n v_1 v_2 \dots v_j e_j$ in C_n^{+++}. Also, any pair of non-adjacent vertices e_k, e_l where $k < l$ and $1 \leq k, l \leq n-1$ has a path $e_k v_k v_{k+1} \dots v_l e_l$ in C_n^{+++}. Also, for any pair of non-adjacent vertices $e_n, e_i \in V(C_n^{+++}) - H$ where $1 \leq i \leq n-1$ there exists a path $e_n v_1 v_2 \dots v_i e_i$ in C_n^{+++}. Hence, H is a hub dominating set with $n-1$ vertices. Therefore, $\gamma_h(C_n^{+++}) = n-1, \forall\, n \geq 3$.

Theorem 3.5

For all $m, n \geq 2, \gamma_h\left(K_{m,n}^{+++}\right) = \min(m,n) + 1$, where $K_{m,n}^{+++}$ is the transformation graph of the complete Bipartite graph $K_{m,n}$.

Proof.

Consider the transformation graph of complete Bipartite graph $K_{m,n}^{+++}$ with $m+n+mn$ vertices. Let $V\left(K_{m,n}^{+++}\right) = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n, e_1, e_2, \dots, e_{mn}\}$, v_i's where $1 \leq i \leq m$ are of degree $2n$; u_j's where $1 \leq j \leq n$ are of degree $2m$ and e_k's where $1 \leq k \leq mn$ are of degree $m+n$.

Case (i): $m \leq n$

We claim $H = \{v_1, v_2, \dots v_m, u_1\}$ is a hub dominating set. Clearly every u_j where $2 \leq j \leq n$ are adjacent to v_1. Also,

$e_1, e_2, \dots, e_n$ are adjacent to v_1,

$e_{n+1}, e_{n+2}, \dots, e_{2n}$ are adjacent to v_2,

$e_{2n+1}, e_{2n+2}, \dots, e_{3n}$ are adjacent to v_3,

$\vdots$

$e_{(m-1)n+1}, e_{(m-1)n+2}, \dots, e_{mn}$ are adjacent to v_m.

Also, any pair of non-adjacent vertices in $V(K_{m,n}^{+++}) - H$ has a path in $K_{m,n}^{+++}$ with the internal vertices in H. Hence, H is a hub dominating set. Therefore,
$\gamma_h(K_{m,n}^{+++}) = m + 1, \forall\, m, n \geq 2$ and $m \leq n$.

Case (ii): $m > n$

We claim $H = \{u_1, u_2, \dots u_n, v_1\}$ is a hub dominating set. Clearly every v_i where $2 \leq i \leq m$ are adjacent to u_1. Also,

$e_1, e_{n+1}, \dots, e_{(m-1)n+1}$ are adjacent to u_1,

$e_2, e_{n+2}, \dots, e_{(m-1)n+2}$ are adjacent to u_2,

$e_3, e_{n+3}, \dots, e_{(m-1)n+3}$ are adjacent to u_3,

$\vdots$

$e_n, e_{2n}, \dots, e_{mn}$ are adjacent to u_n.

Also, any pair of non-adjacent vertices in $V(K_{m,n}^{+++}) - H$ has a path in $K_{m,n}^{+++}$ with the internal vertices in H. Hence, H is a hub dominating set. Therefore,
$\gamma_h(K_{m,n}^{+++}) = n + 1, \forall\, m, n \geq 2$ and $m > n$.

From both the cases, it is clear that $\gamma_h(K_{m,n}^{+++}) = \min(m, n) + 1, \forall\, m, n \geq 2$.

Remark 3.6

$\gamma_h(K_{m,n}^{+++}) = 1$ if $m = 1$ or $n = 1$.

If $n = 1$, then $V(K_{m,1}^{+++}) = \{v_1, v_2, \dots, v_m, u_1, e_1, e_2, \dots, e_{mn}\}$ with $2m + 1$ vertices, where u_1 is of degree $2m$; v_i's where $1 \leq i \leq m$ are of degree 2 and e_k's where $1 \leq k \leq mn$ are of degree $m + n$. Since u_1 is adjacent to all vertices of $K_{m,1}^{+++}$, $\gamma_h(K_{m,1}^{+++}) = 1$. Similarly, $\gamma_h(K_{1,n}^{+++}) = 1$.

Theorem 3.7

For all $n \geq 4, \gamma_h(W_n^{+++}) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even} \\ \frac{n+3}{2} & \text{if } n \text{ is odd} \end{cases}$, where W_n^{+++} is the transformation graph of the Wheel graph W_n.

Proof.

Consider the transformation graph of Wheel graph W_n^{+++} with $3n + 1$ vertices. Let $V(W_n^{+++}) = \{v_1, v_2, \dots, v_{n+1}, e_1, e_2, \dots, e_{2n}\}$, v_i's, e_j's where $1 \leq i, j \leq n$ are of degree 6; v_{n+1} is of degree $2n$ and e_k's where $n + 1 \leq k \leq 2n$ are of degree $n + 3$.

Case (i): n is even

We claim $H = \{v_1, v_3, v_5, \dots v_{n-1}, v_{n+1}\}$ is a hub dominating set. Clearly all v_i's, e_k's $\in V(W_n^{+++}) - H$ where $1 \leq i \leq n$ and $n + 1 \leq k \leq 2n$ are adjacent to v_{n+1}. Also,

e_1, e_n are adjacent to v_1,

e_2, e_3 are adjacent to v_3,

e_4, e_5 are adjacent to v_5,

$\vdots$

e_{n-2}, e_{n-1} are adjacent to v_{n-1}.

Also, any pair of non-adjacent vertices in $V(W_n^{+++}) - H$ has a path in W_n^{+++} with the internal vertices in H. Hence, H is a hub dominating set with $\frac{n}{2} + 1$ vertices. Therefore, $\gamma_h(W_n^{+++}) = \frac{n}{2} + 1$, if n is even.

Case (ii): n is odd

We claim $H = \{v_1, v_2, v_4, \dots v_{n-1}, v_{n+1}\}$ is a hub dominating set. Clearly all v_i's, e_k's $\in V(W_n^{+++}) - H$ where $1 \leq i \leq n$ and $n + 1 \leq k \leq 2n$ are adjacent to $v_{n+1} \in H$. Also,

e_n is adjacent to v_1,

e_1, e_2 are adjacent to v_2,

e_3, e_4 are adjacent to v_4,

$\vdots$

e_{n-2}, e_{n-1} are adjacent to v_{n-1}.

Also, for any pair of non-adjacent vertices in $V(W_n^{+++}) - H$, there exists a path in W_n^{+++} with the internal vertices in H. Hence, H is a hub dominating set. Therefore, $\gamma_h(W_n^{+++}) = \frac{n+3}{2}$, if n is odd.

Hence, $\gamma_h(W_n^{+++}) = \begin{cases} \frac{n}{2}+1 & \text{if } n \text{ is even} \\ \frac{n+3}{2} & \text{if } n \text{ is odd.} \end{cases}$

Remark 3.8

$\gamma_h(W_n^{+++}) = 2$, when $n = 3$.

If $n = 3$, then $V(W_n^{+++}) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3, e_4, e_5, e_6\}$, all v_i's, e_j's where
$1 \le i \le 4$, $1 \le j \le 6$ are of degree 6. We claim $H = \{v_4, e_1\}$ is a hub dominating set. Clearly $v_1, v_2, v_3, e_4, e_5, e_6$ are adjacent to v_4 and e_2, e_3 are adjacent to e_1. Also, any pair of non-adjacent vertices in $V(W_3^{+++}) - H$ has a path in W_3^{+++} with the internal vertices in H. Hence, H is a hub dominating set with 2 vertices. Therefore, $\gamma_h(W_n^{+++}) = 2$, when $n = 3$.

Theorem 3.9

For all $n \ge 3, \gamma_h(H_n^{+++}) = n$, where H_n^{+++} is the transformation graph of the Helm graph H_n.

Proof.

Consider the transformation graph of Helm graph H_n^{+++} with $5n+1$ vertices. Let $V(H_n^{+++}) = \{v_1, v_2, \dots, v_{2n+1}, e_1, e_2, \dots, e_{3n}\}$, v_i's,e_k's where $1 \le i, k \le n$ are of degree 8; v_{n+1} is of degree $2n$; v_j's where $n+2 \le j \le 2n+1$ are of degree 2; e_l's where $n+1 \le l \le 2n$ are of degree $n+4$ and e_m's where $2n+1 \le m \le 3n$ are of degree 5. We claim $H = \{v_1, v_2, v_3, \dots, v_n\}$ is a hub dominating set. Clearly,

v_{n+1}, v_{n+2} are adjacent to v_1,

v_{n+3} is adjacent to v_2,

⋮

v_{2n+1} is adjacent to v_n.

Also,

e_1, e_{n+1}, e_{2n+1} are adjacent to v_1,

e_2, e_{n+2}, e_{2n+2} are adjacent to v_2,

⋮

e_n, e_{2n}, e_{3n} are adjacent to v_n.

Also, any pair of non-adjacent vertices in $V(H_n^{+++}) - H$ has a path in H_n^{+++} with the internal vertices in H. Hence, H is a hub dominating set with n vertices. Therefore, $\gamma_h(H_n^{+++}) = n, \forall\, n \ge 3$.

Conclusion

In this paper, we have found the hub domination number of the transformation graphs $P_n^{+++}, C_n^{+++}, K_{m,n}^{+++}, W_n^{+++}$ and H_n^{+++}. The topics of further research includes the suggestion on finding the bounds on hub domination number of transformation graphs.

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