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On Paired Double Domination number for Degree Splitting Graphs of Path and Cycle related Graphs

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Abstract: In this paper, we introduced the new concept Paired double domination number for degree splitting graph of special graphs. A paired – double dominating set of a graph G with no isolated vertex is a double dominating sets of vertices whose induced subgraph has a perfect matching. A Paired double domination number $\gamma_{prdd}(G)$ is the minimum cardinality of a paired double dominating set of G. Let $G = (V, E)$ be a graph with $V(G) = S_1 \cup S_2 \cup \ldots \cup S_t \cup T$, where S_i is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$, where $1 \leq i \leq t$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \ldots w_t$ and joining w_i to each vertex of S_i for $i = 1,2, \ldots t$ .We establish Paired double domination number for degree splitting graphs of path and cycle related graphs.

MSC : 05C69

Key words: *Double domination number, Paired - double domination number, Locating paired - double domination number.*

1 Introduction

Let G = (V, E) be graph with vertex set V and edge set E. We obtain with some terminology. For a vertex v of a graph G, the open neighbourhood of a vertex $v \epsilon V$ is $N(u) = \{ v/uv \in E \}$ and closed neighborhood of vertex $N[v] = N(v) \cup \{v\}$.

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A subset $S \subseteq V$ is a dominating set of G, if for every vertex $\epsilon V, |N[v] \cap s| \geq 1$. The domination number is the minimum cardinality of a dominating set of G. A subset S of V is double dominating set of G if for every vertex $v \epsilon V, |N[v] \cap s| \geq 2$, that is $v \, is \, in \, S$ and has at least one neighbor in S and v is in V-S has at least two neighbors in S [3].

A set S is called paired – dominating set if it dominates V and < S > contains at least one perfect matching. A paired – dominating set S with matching M is a dominating set $S = \{v_1, v_2, v_3, \ldots, v_{2t-1} v_{2t}\}$ with independent edge set $M=\{e_{1,}e_2,\dots,e_t\}$ where each edge e_j joins two elements of S, that is M is perfect matching of < S > . If $v_jv_k = e_i \in M$, we say that v_{ij} and v_{ik} are paired in S [5]. A set S is called a paired – double dominating set if it is a double dominating set and < S > contains at least one perfect matching. The double domination number $\gamma_{dd}(G)$ is the minimum cardinality of double dominating set of G, the paired – domination number $\gamma_{pr}(G)$ is the minimum cardinality of paired dominating set of G and paired double domination number $\gamma_{prdd}(G)$ is the minimum cardinality of a paired double dominating set of G. $P_n \odot K_{1,m}$is a graph obtained from the path by attaching $K_{1,m}$to each of its vertices A Triangular Snake T_n is obtained from a path $u_1,u_2,u_3,\dots.u_{n-1},u_n$ by joining $u_i$and $u_{i+1}$to a new vertex v_i for $1\le i\le n-1$.That is every edge of a path is replaced by a triangle C_3. An Alternate Triangular Snake $A(T_n)$ is obtained from a path $u_1,u_2,u_3,\dots.u_{n-1},u_n$ by joining $u_i$and u_{i+1} (alternatively) to a new vertex v_i .That is every alternatively edge of a path is replaced by a triangle C_3.

In this paper, we characterize Paired double domination number for degree splitting graphs of $P_n \odot K_1, (P_n \odot K_1)\odot K_{1,m}, P_n \odot K_{1,m}$,Triangular Snake, Alternate Triangular Snake

Theorem 1.1. [7] For any path P_n ,

$$\gamma_{prdd}(P_n) = \begin{Bmatrix} 2 & if n=2 \\ does\ not\ exist & if\ n=3 \\ 2\left\lfloor\frac{n}{3}\right\rfloor+2 & other\ wise \end{Bmatrix}$$

Theorem 1.2. [7] For any cycle C_n $\gamma_{prdd}(C_n)$= $2\left\lceil\frac{n}{3}\right\rceil$

Theorem 1.3. [7] For any path P_n , n≠3 , $\gamma_{dd}(P_n) \le \gamma_{prdd}(P_n)$

Theorem 1.4. [7] For any cycle C_n , $\gamma_{dd}(C_n) \le \gamma_{prdd}(C_n)$

Theorem 1.5. [7] If n = 3k+2 where k ϵ N, then$\gamma_{prdd}(P_n)=\gamma_{prdd}(C_n)$.

2. Main Results

Theorem 2.1

For any integer $n=3$ $\gamma_{prdd}\big(DS(P_3\odot K_1)\big)=6$.

Proof:

Let $\{v_1,v_2,v_3,u_1,u_2,u_3\}$ be the vertices of $P_3\odot K_1\ where\ S_1=\{u_1,u_2,u_3\}$ and $S_2=\{v_1,v_3\}$ are sets of values of deg 1 and deg 2 respeertively. To obtain $DS(P_3\odot K_1)$ from$P_3\odot K_1$, we add $x_1$and x_2 which corresponds to $S_1$and S_2 respectively. Let H be the γ_{prdd} set of $DS(P_3\odot K_1)$. Then $H=\{v_1,v_2,v_3,x_1,x_2,u_3\}$ and $<H>$ has a perfect matching. Hence $\gamma_{prdd}\big(DS(P_3\odot K_1)\big)=6$.

Theorem 2.2

For any integer $n\ge 4$ $\gamma_{prdd}\big(DS(P_n\odot K_1)\big)=\begin{Bmatrix} n+3 & if\ n\ is\ odd \\ n+2 & if\ n\ is\ even\end{Bmatrix}$.

Proof:

Let $\{v_1,v_2,\dots,v_n,u_1,u_2,\dots,u_n\}$ be the vertices of $P_n\odot K_1$ with partition $S_1=\{u_1,u_2,u_3,\dots.u_{n-1},u_n\}$ $S_2=\{v_1,v_n\}$ and $S_3=\{v_2,v_3,\dots.v_{n-1}\}$. To obtain $DS(P_n\odot K_1)$ from $P_n\odot K_1$, we add x_1 , x_2 and x_3 which corresponds to S_1 , S_2 and S_3 respectively. As a result

$$V\big(DS(P_n\odot K_1)\big)=\{x_1,x_2,x_3,v_1,v_2,v_3,\dots.v_{n-1},v_n,u_1,u_2,u_3,\dots.u_{n-1},u_n\}$$

where $|V(DS(P_n \odot K_1))| = 2n + 3$ for $n \geq 4$. Let H be the γ_{prdd} set of $DS(P_n \odot K_1)$ and let $H = \{v_1, v_2, v_3, \dots . v_{n-1}, v_n, u_1, x_1\}$ and $H_2 = \{x_2\}$.

Case(i) n is odd

Let $H = H_1 \cup H_2$. Then $< H >$ contains $a\, P_{n+3}$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}(DS(P_n \odot K_1)) = n + 3$.

Case(ii) n is even

Let $H = H_1$. Then $< H >$ contains $a\, P_{n+2}$ graph and $< H >$ has a perfect matching. Hence

$\gamma_{prdd}(DS(P_n \odot K_1)) = n + 2$.

Theorem 2.3

For any integer $n = 3$ $\gamma_{prdd}(DS((P_3 \odot K_1) \odot K_{1,n})) = 8$.

Proof:

Let $\{v_1, v_2, v_3, u_1, u_2, u_3, u_{11}, u_{12}, \dots u_{31}, u_{32}, , \dots u_{3,n}\}$ be the vertices of $(P_3 \odot K_1) \odot K_{1,n}$ *where* $S_1 = \{v_1, v_3\}$, $S_2 = \{u_1, u_2, u_3\}$ and $S_3 = \{u_{1,1}, u_{1,2}, u_{3,1}, u_{3,2}, , \dots u_{3,n}\}$ are sets of vertices of deg 2 , deg n+1 and deg 1 respectively. To obtain $DS((P_3 \odot K_1) \odot K_{1,n}$ from $P_3 \odot K_1$, we add x_1, x_2 and x_3 which corresponds to S_1, S_2 and S_3 respectively. Let H be the γ_{prdd} set of $DS((P_3 \odot K_1) \odot K_{1,n})$. Then $H = \{v_1, v_2, v_3, u_1, u_2, u_3, x_3, u_{3,n}\}$ and $< H >$ has a perfect matching. Hence$\gamma_{prdd}(DS((P_3 \odot K_1) \odot K_{1,n}) = 8$.

Theorem 2.4

For any integer $n \geq 4$ $\gamma_{prdd}(DS((P_n \odot K_1) \odot K_{1,m})) = 2n + 2$.

Proof:

Let

$$\{v_1, v_2, \dots, v_n, u_1, u_2, \dots ., u_n, u_{1,1}, u_{1,2}, . . u_{n,1}, u_{n2}, , \dots u_{n,m}\}$$

be the vertices of $(P_n \odot K_1) \odot K_{1,n}$ with partition $S_1 = \{u_{1,1}, u_{1,2}, \dots, u_{n,1}, u_{n,2}, \dots u_{n,m}\}$, $S_2 = \{u_1, u_2, \dots ., u_n\}$, $S_3 = \{v_1, v_n\}$ and $S_4 = \{v_2, v_3, \dots . v_{n-1}\}$. To obtain $DS((P_3 \odot K_1) \odot K_{1,n}$ from $(P_n \odot K_1) \odot K_{1,n}$, we add x_1, x_2 , x_3 and x_4 which corresponds to S_1, S_2 ,S_3 and S_4 respectively. As a result,$DS\left((P_n \odot K_1) \odot K_{1,m}\right) = \{x_1, x_2, x_3, x_4, v_1 \dots, v_n, u_1, u_2, \dots ., u_n, u_{1,1}, u_{1,2}, . . u_{n,1}, u_{n,2}, , \dots u_{n,m}\}$ .Let H be the γ_{prdd} set of $DS((P_n \odot K_1) \odot K_{1,m})$. Then $H = \{x_1, u_{n,m}, v_1, . . , v_n, u_1, u_2, \dots ., u_n\}$. and $< H >$ has a perfect matching. Hence$\gamma_{prdd}(DS((P_n \odot K_1) \odot K_{1,m}) = 2n + 2$.

Theorem 2.5

For any integer $n = 3$ $\gamma_{prdd}\left(DS(P_3 \odot K_{1,n})\right) = 6$.

Proof:

Let $\{v_1, v_2, v_3, u_{1,1}, u_{1,2}, \dots u_{3,1}, u_{3,2}, , \dots u_{3,n}\}$ be the vertices of $(P_3 \odot K_{1,n})$ *where* $S_1 = \{u_{1,1}, u_{1,2}. . , u_{3,1}, u_{32}, , \dots u_{3n}\}$ and $S_2 = \{v_1, v_3\}$ are sets of vertices of deg 1 and deg n+1 respectively. To obtain $DS(P_3 \odot K_{1,n})$ from $P_3 \odot K_{1,n}$, we add x_1 and x_2

which corresponds to S_1 and S_2 respectively. Let H be the γ_{prdd} set of $DS(P_3 \odot K_{1,n})$. Then $H = \{v_1, v_2, v_3, x_1, x_2, u_{3,n}\}$ and $< H >$ has a perfect matching. Hence $\gamma_{prdd}\left(DS(P_3 \odot K_{1,n})\right) = 6$.

Theorem 2.6

For any integer $n \geq 4$ $\gamma_{prdd}\left(DS(P_n \odot K_{1,m})\right) = \begin{Bmatrix} n+3 & if\ n\ is\ odd \\ n+2 & if\ n\ is\ even \end{Bmatrix}$.

Proof:

Let $\{v_1, v_2, \dots, v_n, u_{1,1}, u_{,12}, .. u_{n,1}, u_{n,2}, , \dots u_{n,m}\}$ be the vertices of $(P_n \odot K_{1,m})$ $with\ partition\ S_1 = \{u_{1,1}, u_{1,2}, .. u_{n,1}, u_{n,2}, , \dots u_{n,m}\}$, $S_2 = \{v_1, v_n\}$ and $S_3 = \{v_2, v_3, \dots . v_{n-1}\}$. To obtain $DS(P_n \odot K_{1,m})$ from $(P_n \odot K_{1,m})$, we add x_1, x_2 and x_3 which corresponds to S_1, S_2 and S_3 respectively. As a result, $DS(P_n \odot K_{1,m}) = \{x_1, x_2, x_3, v_1 \dots, v_n, u_{1,2}, .. u_{n,1}, u_{n,2}, , \dots u_{n,m}\}$. Let H be the γ_{prdd} set of $DS(P_n \odot K_{1,m})$. Then $H_1 = \{x_1, u_{1,1}, v_1, .., v_n, u_1, u_2, \dots ., u_n\}$ and $H_2 = \{x_1\}$.

Case(i) n is odd

Let $H = H_1 \cup H_2$. Then $< H >$ contains $a\ P_{n+3}$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}\left(DS(P_n \odot K_{1,m})\right) = n + 3$.

Case(i) n is even

Let $H = H_1$. Then $< H >$ contains $a\ P_{n+2}$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}\left(DS(P_n \odot K_{1,m})\right) = n + 2$.

Theorem 2.7

For any integer $n \geq 3$ $\gamma_{prdd}(DS(AT_n)) = \begin{Bmatrix} 4\left(\frac{n}{3}\right) + 2 & if\ n \equiv 0\ (mod\ 3) \\ 4\left\lfloor\frac{n}{3}\right\rfloor + 2 & if\ n \equiv 1\ (mod\ 3) \\ 4\left\lceil\frac{n}{3}\right\rceil + 2 & if\ n \equiv 2\ (mod 3) \end{Bmatrix}$.

Proof:

Let $\{v_1, v_2, v_3, \dots . v_{2n}, u_1, u_2, \dots . u_{n-1}, u_n\}$ be the vertices of AT_n with partition $S_1 = \{v_1, v_n, u_1, u_2, \dots . u_{n-1}, u_n\}$ and $S_2 = \{v_2, v_3, \dots . v_{n-1}\}$. To obtain $DS(AT_n)$ from AT_n, we add x_1, and x_2 which corresponds to S_1 and S_2 respectively. As a result, $DS(AT_n) = \{x_1, x_2, v_1, v_2, v_3, \dots . v_{2n}, u_1, u_2, \dots . u_{n-1}, u_n\}$ where $|V\ (DS(AT_n))| = 3n + 2$ for $n \geq 3$. Let $H_1 = \{v_i, i \equiv 1,2\ (mod 3)\}$, $H_2 = \{v_{n-1}\}$ and $H_3 = \{x_1\}$.

Case(i) $n \equiv 0\ (mod 3)$

Then $H = H_1 \cup H_2 \cup H_3$. Thus $< H >$ contains $a\ P_6$ graph and $\left(\frac{2n-6}{3}\right) P_2$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}(DS(AT_n)) = 6 + 2\left(\frac{2n-6}{3}\right) = 4\left(\frac{n}{3}\right) - 4 + 6 = 4\left(\frac{n}{3}\right) + 2$.

Case(ii) $n \equiv 1\ (mod 3)$

Then $H = H_1 \cup H_3$.Thus $< H >$ contains $\left(\frac{2n+1}{3}\right) P_2$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}(DS(AT_n)) = 2\left(\frac{2n+1}{3}\right) = 4\left(\frac{n}{3}\right) + \frac{2}{3} = 4\left(\frac{n}{3}\right) + \frac{2}{3} - \frac{4}{3} + \frac{4}{3} = 4\left(\frac{n-1}{3}\right) + \frac{6}{3} = 4\left\lfloor\frac{n}{3}\right\rfloor + 2$.

Case(iii) $n \equiv 2\ (mod 3)$

Then $H = H_1 \cup H_3$.Thus $< H >$ contains $a\, P_4$ graph $\left(\frac{2n-4}{3}\right) P_2$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}\big(DS(AT_n)\big) = 4 + 2\left(\frac{2n-4}{3}\right) = 4\left(\frac{n}{3}\right) + 4 - \frac{8}{3} = 4\left(\frac{n+1}{3}\right) + 2 = 4\left\lceil\frac{n}{3}\right\rceil + 2.$

Theorem 2.8

For any integer $n = 2$ $\gamma_{prdd}\big(DS(T_2)\big) = 4$.

Proof:

Let $\{v_1, v_2, v_3, u_1, u_2\}$ be the vertices of T_2 *where* $S_1 = \{v_1, v_3, u_1, u_2\}$ are sets of vertices of deg 1. To obtain $DS(T_2)$ from T_2 we add x which corresponds to S respectively. Let H be the γ_{prdd} set of $DS(T_2)$. Then $H = \{v_1, v_2, v_3, x\}$ and $< H >$ has a perfect matching. Hence $\gamma_{prdd}(DS(T_2)) = 4$.

Theorem 2.9

For any integer $n \geq 3$ $\gamma_{prdd}\big(DS(T_n)\big) =$

$$\left\{\begin{matrix} 2\left(\frac{n}{3}\right) + 2 \quad if\ n \equiv 0\ (mod\ 3) \\ 2\left\lceil\frac{n}{3}\right\rceil + 2 \quad if\ n \equiv 1\ (mod\ 3) \\ 2\left\lceil\frac{n}{3}\right\rceil + 2 \quad if\ n \equiv 2\ (mod 3) \end{matrix}\right\}.$$

Proof:

Let $\{v_1, v_2, v_3, \ldots . v_{n+1}, u_1, u_2, \ldots . u_{n-1}, u_n\}$ be the vertices of T_n with partition $S_1 = \{v_1, v_{n+1}, u_1, u_2, \ldots . u_{n-1}, u_n\}$ and $S_2 = \{v_2, v_3, \ldots . v_{n-1}\}$. To obtain $DS(T_n)$ from T_n, we add x_1 , and x_2 which corresponds to S_1 and S_2 respectively. As a result, $DS(T_n) = \{x_1, x_2, v_1, v_2, \ldots . v_{n+1}, u_1, u_2, \ldots u_n\}$ where $|V\ DS(T_n)| = 2n + 1$ for $n \geq 3$. *Let* $H_1 = \{v_i, i \equiv 1,2\ (mod3)\}$, $H_2 = \{v_n\}$ and $H_3 = \{x_1\}$.

Case(i) $n \equiv 0\ (mod3)$

Then $H = H_1 \cup H_3$. Thus $< H >$ contains $a\, P_4$ graph and $\left(\frac{n-3}{3}\right) P_2$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}\big(DS(T_n)\big) = 4 + 2\left(\frac{n-3}{3}\right) = 2\left(\frac{n}{3}\right) - 2 + 4 = 2\left(\frac{n}{3}\right) + 2.$

Case(ii) $n \equiv 1\ (mod3)$

Then $H = H_1 \cup H_3$.Thus $< H >$ contains $\left(\frac{n-1}{3} + 1\right) P_2$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}\big(DS(T_n)\big) = 2\left(\frac{n-1}{3} + 1\right) = 2\left(\frac{n-1}{3}\right) + 2 = 2\left\lceil\frac{n}{3}\right\rceil + 2.$

Case(iii) $n \equiv 2\ (mod3)$

Then $H = H_1 \cup H_2 \cup H_3$. Thus $< H >$ contains $a\, P_6$ graph $\left(\frac{n-5}{3}\right) P_2$ graph and $< H >$ has a perfect matching. Hence $\gamma_{prdd}\big(DS(T_n)\big) = 6 + 2\left(\frac{n-5}{3}\right) = 2\left(\frac{n-5}{3} + 2\right) + 2 = 2\left(\frac{n+1}{3}\right) + 2 = 2\left\lceil\frac{n}{3}\right\rceil + 2.$

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